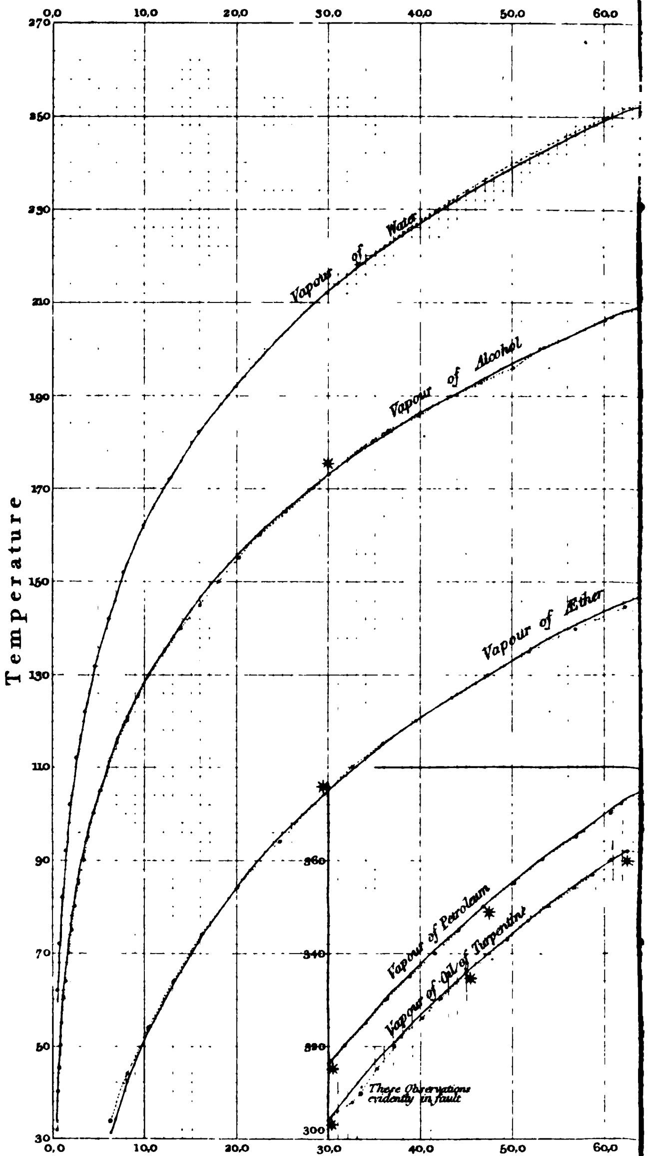

The Abscissa represents the Pressure in

Calculated

The Observations

ON

THE HEAT OF VAPOURS

AND ON

ASTRONOMICAL REFRACTIONS.

BY

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PREFACE.

THE connexion between the temperature and the pressure (or elasticity) of elastic vapours is a desideratum in Physics. A knowledge of it is indispensable to an exact theory of the Steam Engine, to an exact theory of Astronomical Refractions, and to an accurate solution of other important problems. The want of it has hitherto been supplied by unsatisfactory approximations; but these questions cannot be completely investigated without a more careful attention to the premises than has hitherto been possible, owing to a want of the proper key to these researches, which consists in a knowledge of the mathematical law which connects the temperature and the pressure in elastic fluids, and which is required in addition to the law of Mariotte and Gay Lussac to complete their theory.

If V represent the absolute heat or *caloric*, i the *latent* heat, c the *sensible* heat or that which affects the *thermometer*,

$$V = i + c.$$

If θ be the *temperature* as indicated by a thermometer, there can be little doubt that V is capable of being expressed in a series proceeding according to positive powers of θ, so that

$$V = a + b\theta + c\theta^2 + \&c.$$

a, b, c, &c., have a certain signification in Taylor's theorem, but

without being able to determine their values, *à priori*, or to obtain any relations between them, they may be treated as constants. If the *latent* heat be constant, which is probable, and if the effect indicated by the thermometer is proportional to the sensible heat,

$$c = b\,\theta, \quad V = a + b\,\theta.$$

It must, however, be left to experiment to decide how many terms are to be taken into account for any given substance, within any given range of the thermometric scale, and in order to satisfy the results of observation within any given quantity. The other suppositions upon which my theory is founded are those of Laplace, viz. that the quantity called γ by M. Poisson is constant for the same substance at different temperatures, and that the equation

$$V = A + B\frac{p^{\frac{1}{\gamma}}}{\varrho}$$

is the solution of a certain differential equation. See *Méc. Cél.*, vol. v. p. 108. Poisson, *Méc.*, vol. ii. p. 640.

The theorems which are given by M. Poisson in the second volume of the *Mécanique*, and which are also to be found in the works of Pouillet and Navier, rest upon the condition that the absolute heat is constant, while the sensible heat varies. This is the most restricted hypothesis which can be made upon the nature of heat, and it does not satisfy the observations. In this Treatise I have gone a step further, by supposing the absolute heat to vary with the sensible heat, or to be represented by an expression of the form $a + b\,\theta$, (or what is the same, $V = C + D\,(1 + \alpha\,\theta)$. See p. 2.) θ being the temperature reckoned from some fixed point, a and b constants. This includes implicitly the other hypothesis, which if true, in determining a and b by means of observations, the constant b should come out zero. This in the case of steam is certainly not the case, nor is it so in any case which I have examined.

The experiments of Dulong and Arago upon steam at high temperatures, those of Southern and Dalton, and those of Dr. Ure, furnish data by which the supposition I have adopted and the formulæ which flow from it can be scrutinized; and if the expressions which result from it fail to represent those observations, we have at least arrived at this conclusion, that the condition of the invariability of the quantity called γ by M. Poisson does not obtain in nature, or that the absolute heat cannot be represented by so simple a function of the temperature or sensible heat. Recourse must then be had to more complicated expressions. If, on the contrary, my formula represents the observations of the temperature of vapours with accuracy, its origin in a simple theoretical notion of the quantity of absolute heat, and its simplicity, are great additional recommendations in its favour. The formula which I have obtained does, I believe, represent the observations better than any hitherto devised; at low temperatures and pressures it deviates a little, but a very slight error in the observed pressures may account for this discrepancy. Dalton says that it is next to impossible to free any liquid entirely from air; of course if any air enter, it unites its force to that of the vapour. Moreover, when the pressures are small, the variation of temperature becomes great for a small variation of pressure; so that the agreement of theory with observation may be considered as complete, even if the absolute amount of the error of the calculated temperature is then more considerable.

My formula has also been compared with the observations of Dr. Ure, on the vapour of alcohol, æther, petroleum, and oil of turpentine, recorded in the Philosophical Transactions for 1818.

I think that the comparisons contained in this treatise afford sufficient evidence that my formula is established, and that the deviations of the calculated results from those of observation are within the limits of the errors of the latter; but this point I leave to be decided by those more conversant with the nature of the experi-

ments. It would not militate against my views if it were found necessary to take in an additional term and to make

$$V = C + D(1 + \alpha\theta) + E(1 + \alpha\theta)^2 + \&c.$$

but the expressions for the temperature and density in terms of the pressure would not be quite so simple, although more pliable.

As the same principles must be applicable to the constitution of the atmosphere, I have examined the observations made by M. Gay Lussac in his aëronautic ascent from Paris, and which are published by M. Biot in the *Conn. des Temps.* My calculated temperatures may be considered as identical with the *températures regularisées* of M. Biot, which are given by that distinguished philosopher as representing the condition of the atmosphere divested of the irregularities and errors incidental to observations made under circumstances so difficult and so disadvantageous. But the altitude to which man can ascend is so limited, that observations of the temperature made in aëronautic ascents will never furnish so complete a test of the accuracy of any formula professing to give the relation between the pressure and the temperature in elastic fluids, as observations of the temperature of the vapour of water and other substances, which can be carried through a greater range of the thermometric scale, and above all through the low pressures where the character of the curve is more decided.

M. Biot has dwelt with reason upon the importance of introducing into the theory of Astronomical Refractions a greater conformity with the conditions of the problem than has hitherto been attempted; and he has also noticed the imperfection in principle of the present mode of calculating heights by observations of the barometer, a method which must of course be abandoned (at least in any accurate exposition of this theory) whenever the discovery of the true connexion between the temperature and the pressure of the higher regions of the atmosphere renders it possible to adopt a more rigorous mode of eliminating the density from the differ-

ential equation which connects d p and d z. The correct expression which connects the difference of altitude with the pressures at the upper and lower stations ought to be the foundation of the theory of Refractions. Considering on the one hand the notions upon which my formula is ultimately founded, its identity with the results offered by the observations of steam and other vapours, and moreover the agreement afforded by the direct comparison with the observations of M. Gay Lussac, there can be no doubt that it represents the density of the atmosphere at different altitudes with greater fidelity than any hypothesis which has up to the present time been made the basis of the theory of Astronomical Refractions.

I think that my table of mean refractions represents the observed quantities within the limits of their probable errors, and I have obtained this result without any arbitrary alterations of the constants.

In the higher regions of the atmosphere the cold is intense*, depriving the air of its elasticity and converting it into a liquid or solid substance. My formula of course is only applicable as long as the air continues in the state of an elastic vapour; and if at any altitude it ceases to maintain that condition, the density must be represented by a discontinuous function. But the density of this frozen air must be extremely small, and it probably has little effect upon the amount of Refraction.

I am indebted to Mr. Russell for his kind assistance in the numerical calculations which accompany this treatise.

* See Poisson, *Théorie de la Chaleur*, p. 460.

29, *Eaton Place*,
March 2, 1840.

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ON

THE HEAT OF VAPOURS

AND ON

ASTRONOMICAL REFRACTIONS.

GENERAL EXPRESSIONS.

LET V be the quantity of absolute heat, considered as a function of the sensible heat or temperature θ,

$$\frac{dV}{d\theta} = \frac{dV}{d\varrho}\frac{d\varrho}{d\theta} + \frac{dV}{dp}\frac{dp}{d\theta}$$

$$p = k\,\varrho\,(1 + \alpha\,\theta).$$

ϱ being the density, p the pressure, k and α constants,

$$\frac{d\varrho}{d\theta} = -\frac{\alpha\,\varrho}{1 + \alpha\,\theta}$$

$$\frac{dp}{d\theta} = \frac{\alpha\,p}{1 + \alpha\,\theta}$$

if

$$-\frac{dV}{d\varrho}\,\frac{\alpha\,\varrho}{(1 + \alpha\,\theta)} = \gamma\,\frac{dV}{dp}\,\frac{\alpha\,p}{(1 + \alpha\,\theta)}$$

$$\varrho\,\frac{dV}{d\varrho} + \gamma p\,\frac{dV}{dp} = 0.$$

If γ be considered as a constant quantity the integral of this partial differential equation is

$$\frac{p^{\frac{1}{\gamma}}}{\varrho} = \text{funct}^{\text{n}}.\ V*.$$

The simplest form which can be assigned to this function of V is such that

$$V = A + B\,\frac{p^{\frac{1}{\gamma}}}{\varrho}$$

A and B being constants.

* So far the reasoning is identical with that contained in the *Mécanique* of M. Poisson, but M. Poisson proceeds further upon the limited supposition of V being constant.

Laplace arrived at this equation (*Méc. Cél.* vol. v. p. 128.).

See Poisson, *Annales de Chimie*, tom. xxiii. p. 342; *Mécanique*, vol. ii. p. 648; Navier, *Leçons données à l'Ecole des Ponts et Chaussées*, tom. ii.

$$V = A + B \frac{p}{\varrho}^{\frac{1}{\gamma}} \qquad V' = A + B \frac{p'}{\varrho'}^{\frac{1}{\gamma}}$$

$$p = k\varrho\,(1 + \alpha\,\theta) \qquad p' = k\,\varrho'\,(1 + \alpha\,\theta')$$

k and α being constants.

I will now introduce the additional condition that the heat is *proportional* to the temperature, in which case

$$V = C + D(1 + \alpha\,\theta)$$
$$V' = C + D\,(1 + \alpha\,\theta')$$

C and D being constants. These equations include implicitly the hypothesis attributed to Watt and also that of Southern, respecting the vapour of water: on the former $D = 0$. Hence

$$V = C + D\,(1 + \alpha\,\theta) = A + B\,\frac{p}{\rho}^{\frac{1}{\gamma}}$$

$$V' = C + D\,(1 + \alpha\,\theta') = A + B\,\frac{p'}{\rho'}^{\frac{1}{\gamma}}$$

$$1 + \alpha\,\theta' = (1 + \alpha\,\theta)\,\frac{p'^{\frac{\gamma-1}{\gamma}}}{p^{\frac{\gamma-1}{\gamma}}}\left\{\frac{1 - \frac{D}{k\,B}\,p^{\frac{\gamma-1}{\gamma}}}{1 - \frac{D}{k\,B}\,p'^{\frac{\gamma-1}{\gamma}}}\right\}$$

If $\frac{D}{k\,B} = E$ and if θ correspond to the boiling point, $\theta = 180°$ in Fahrenheit's scale, if the pressure be measured in atmospheres $p = 1$, but generally

$$1 + \alpha\,\theta' = (1 + \alpha\,\theta)\,\frac{(p^{\frac{1-\gamma}{\gamma}} - E)^{*}}{(p'^{\frac{1-\gamma}{\gamma}} - E)} \qquad [1]$$

* This equation must not be confounded with another equation which may be deduced from it by making $E = 0$, and which is not reconcileable with phenomena, as was long since noticed by M. Poisson in the case of steam. An equation of this kind is given by M. Pouillet in the form

$$s = \cdot 2669\left(\frac{760}{p}\right)^{1 - \frac{1}{1\cdot 375}}$$

as $$p' = k\varrho'(1+\alpha\theta')$$

$$\frac{\varrho'}{\varrho} = \frac{p'\left(p'^{\frac{1-\gamma}{\gamma}} - E\right)}{p\left(p^{\frac{1-\gamma}{\gamma}} - E\right)}, \qquad [2]$$

$$= \left(\frac{p'}{p}\right)^{\frac{1}{\gamma}} \left\{\frac{1 - E p'^{\frac{\gamma-1}{\gamma}}}{1 - E p^{\frac{\gamma-1}{\gamma}}}\right\}$$

if $$\frac{E p^{\frac{\gamma-1}{\gamma}}}{1 - E p^{\frac{\gamma-1}{\gamma}}} = -H$$

$$\left(\frac{p'}{p}\right)^{\frac{\gamma-1}{\gamma}} = 1 - q \qquad \frac{1 - E p'^{\frac{\gamma-1}{\gamma}}}{1 - E p^{\frac{\gamma-1}{\gamma}}} = 1 - Hq$$

$$\frac{\varrho'}{\varrho} = (1-q)^{\frac{1}{\gamma-1}}(1 - Hq),$$

if $$\log(1 - Hq) = -u \qquad c^{-u} = 1 - Hq$$

c being the number of which the hyperbolic logarithm equals unity

$$\frac{\varrho'}{\varrho} = H^{\frac{1}{1-\gamma}} c^{-u}\left\{c^{-u} - 1 + H\right\}^{\frac{1}{\gamma-1}}.$$

if $\frac{\varrho'}{\varrho} = 1 - \omega$

$$\omega = 1 - H^{\frac{1}{1-\gamma}} c^{-u}\left\{c^{-u} - 1 + H\right\}^{\frac{1}{\gamma-1}}.$$

Since $C + D(1+\alpha\theta) = A + B\frac{p}{\varrho}^{\frac{1}{\gamma}}$.

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for atmospheric air. *Elémens de Physique*, vol. i. p. 400, and by Navier, *Leçons données à l'Ecole des Ponts et Chaussées*, vol. ii. p. 310, in the form

$$v' = \frac{(1+\alpha v)}{\alpha}\left(\frac{\varpi'}{\varpi}\right)^{\cdot 3748} - \frac{1}{\alpha}.$$

If $\frac{1}{\varrho} = v$

$$\alpha\, D\,(\theta' - \theta) = B\, p^{\frac{1}{\gamma}}\, \{v' - v\} = V' - V,$$

supposing the heat and the volume to vary, the pressure remaining constant.

According to Dulong the following laws obtain, which however, are not admitted by Dr. Apjohn (see Phil. Mag. 1838, p. 339):

"1°. Des volumes égaux de tous les fluides élastiques pris à une même température et sous une même pression, étant comprimés ou dilatés subitement d'une même fraction de leur volume, dégagent ou absorbent la même quantité absolue de chaleur. 2°. Les variations de température qui en résultent sont en raison inverse de leur chaleur spécifique à volume constant."—*Mém. de l'Institut*, tom. x. p. 188.

According to the first of these laws the quantity B must be the same for different vapours; of the second I am unable to offer any satisfactory interpretation.

In what follows I propose to ascertain how far the equations [1] and [2] satisfy the best observations on record. The general relation gives

$$1 + \alpha\,\theta'' = (1 + \alpha\,\theta)\,\frac{\left(p^{\frac{1-\gamma}{\gamma}} - E\right)}{\left(p''^{\frac{1-\gamma}{\gamma}} - E\right)}.$$

Eliminating E between this equation and that which connects θ' and p',

$$(\theta'' - \theta)\,(1 + \alpha\,\theta')\left(p'^{\frac{1-\gamma}{\gamma}} - p^{\frac{1-\gamma}{\gamma}}\right) = (\theta' - \theta)\,(1 + \alpha\,\theta'')\left(p''^{\frac{1-\gamma}{\gamma}} - p^{\frac{1-\gamma}{\gamma}}\right)$$

If $\frac{1-\gamma}{\gamma} = \beta$

$$\frac{\left(\frac{p''}{p}\right)^{\beta} - 1}{\left(\frac{p'}{p}\right)^{\beta} - 1} = \frac{(\theta'' - \theta)\left(\frac{1}{\alpha} + \theta'\right)}{(\theta' - \theta)\left(\frac{1}{\alpha} + \theta''\right)}.$$

From this equation, knowing θ'', θ', θ, p'', p', p; β may be determined for any gas or vapour. Knowing β, E may be found from the equation

$$E = \frac{p'^{\beta}\left(\frac{1}{\alpha} + \theta'\right) - p^{\beta}\left(\frac{1}{\alpha} + \theta\right)}{\theta' - \theta}.$$

ON THE PRESSURE OF STEAM.

The most accurate and extensive experiments by which the accuracy of these relations can be tested are those which have been made upon the conditions of steam. The following are the experiments of Arago and Dulong, as recorded in tom. x. of the *Mémoires de l'Institut*, p. 231; together with the temperatures calculated by the best empirical formulæ.

Nos. des observations.	Elasticité en mètres de mercure à 0°.	Elasticité en atmosph. de 0m·76.	Température observée.	Température calculée par la formule de Tredgold.	Température calculée par la formule de Roche cöeff. moyen.	Température calculée par la formule de Coriolis.	Température calculée par la formule adoptée.
			Cent.	Cent.	Cent.	Cent.	Cent.
1	1·62916	2·14	123°·7	123°·54	123°·58	123°·45	122°·97
3	2·1816	2·8705	133·3	133·54	133·43	133·34	132·9
5	3·4759	4·5735	149·7	150·39	150·23	150·3	149·77
8	4·9383	6·4977	163·4	164·06	163·9	164·1	163·47
9	5·6054	7·3755	168·5	169·07	169·09	169·3	168·7
15	8·840	11·632	188·5	188·44	188·63	189·02	188·6
21	13·061	17·185	206·8	206·15	207·04	207·43	207·2
22	13·137	17·285	207·4	206·3	206·94	207·68	207·5
25	14·0634	18·504	210·5	209·55	210·3	211·06	210·8
28	16·3816	21·555	218·4	216·29	218·01	218·66	218·5
30	18·1894	23·934	224·15	222·09	233·4	224·0	224·02

There are reasons which make it probable that in inquiries of this nature the scale of temperature as indicated by the expansion of air is to be preferred, although the difference between the indications of a mercury thermometer with that of air is not considerable.

The following table is given by M. Pouillet (*Elémens de Physique*, vol. i. p. 259) for the centigrade scale:

Températures indiquées par le therm. à mercure, à enveloppe de verre.	Températures indiquées par un therm. à air, et corrigées de la dilatation du verre.	Volumes correspondans d'une même masse d'air.
− 36°	− 36°	0·8650
0	0	1·0000
100	100	1·3750
150	148·70	1·5576
200	197·05	1·7389
250	245·05	1·9189
300	292·70	2·0976
Ebull. du merc. 360	350·00	2·3125

From the above I have deduced the following Table for Fahrenheit's scale:

Merc. therm.	Air therm.	Merc. therm.	Air therm.
212°	212°	482°	478°·1
302	299·7	572	558·9
392	386·7	680	662·0

I now proceed to determine for steam the constants γ and E by means of the observations of Dulong and Arago which I have quoted in p. 5.

For the air thermometer on Fahrenheit's scale the experiments of Dulong and Arago (*Mém. de l'Institut*, vol. x.) give, θ being reckoned in Fahrenheit's scale and from the freezing point of water,

$$p = 1 \qquad \theta = 180 \qquad \frac{1}{\alpha} = 480^\circ$$

$$p' = 11{\cdot}632 \qquad \theta' = 334{\cdot}7 \qquad \frac{1}{\alpha} + \theta' = 814{\cdot}7$$

$$p'' = 23{\cdot}934 \qquad \theta'' = 396{\cdot}4 \qquad \frac{1}{\alpha} + \theta'' = 876{\cdot}4.$$

I find from these observations

$$\frac{p''^{\beta} - 1}{p'^{\beta} - 1} = [0{\cdot}1140623],$$

the quantity within brackets being the logarithm of the corresponding number; and hence I find

$$\beta = {\cdot}0134^{*} \qquad \gamma = {\cdot}98677 \qquad \frac{1}{\gamma} = 1{\cdot}0134$$

$$E = 1{\cdot}17602 \qquad \log E = {\cdot}0704184 \qquad H = 6{\cdot}6809.$$

The pressure at the boiling point of water (212°) being unity,

$$\frac{1}{\alpha} + \theta = -\frac{[2{\cdot}0651059]}{p^{{\cdot}0134} - 1{\cdot}17602};$$

so that if τ is the number of degrees on Fahrenheit's scale of the air thermometer, and the pressure p be reckoned in atmospheres,

* This value of β appears to me to be the only one which will satisfy the equation.

$$\tau = -\frac{[2{\cdot}0651059]}{p^{\cdot 0134} - 1{\cdot}17602} - 448^{\circ},$$

and if ϱ be the density of steam, the relative volume

$$\frac{\varrho}{\varrho'} = \frac{p\,\{p^{\cdot 0134} - 1{\cdot}17602\}}{p'\,\{p'^{\cdot 0134} - 1{\cdot}17602\}}.$$

In order to ascertain how far the new expression here given for τ represents the totality of the observations, I have calculated the temperatures corresponding to all the observed pressures in the observations of Arago and Dulong, and the results are exhibited in the following table.

Pressure in atmospheres.	Temperature.				Error of temperature calculated by Lubbock. Fahr.
	Observed.			Calculated.	
	Merc. therm. Cent.	Merc. therm. Fahr.	Air therm. Fahr.	Air therm. Fahr.	
2·1400	123°·7	254°·66	253°·6	252°·8	−°·8
2·8705	133·3	271·94	270·4	270·1	−·3
4·5735	149·7	301·46	299·2	299·4	+·2
6·4977	163·4	326·12	323·0	323·2	+·2
7·3755	168·5	335·30	331·9	332·3	+·4
11·6320	188·5	371·30	366·7*	366·7	0
17·1850	206·8	404·24	398·6	398·9	+·3
17·2850	207·4	405·32	399·5	399·4	−·1
18·5040	210·5	410·90	404·9	405·3	+·4
21·5550	218·4	425·12	418·5	418·8	+·3
23·9340	224·15	435·47	428·4*	428·5	+·1

The observations marked with an asterisk were employed in determining the constants.

The error of the temperature calculated by the formula adopted by Arago and Dulong corresponding to the first observation is $-\cdot 73$ cent. or $-1^{\circ}\cdot 3$ of Fahr. I have no doubt that the observed temperature is in excess, and the agreement with the rest of the observations is so complete that within this range of temperature the formula may, I think, be considered as exactly representing the phenomena. The errors of the temperatures, calculated by the various empirical expressions which have been hitherto proposed, are much greater, as may be seen in the table of Dulong and Arago. The following observations are those of Southern, given in p. 172, vol. ii., of Dr. Robison's Mechanical Philosophy.

Pressure.	Temperature.		Error of calculated temp.	Pressure.	Temperature.		Error of calculated temp.
	Observed.	Calculated.			Observed.	Calculated.	
Inch.		°	°	Inch.			°
·52	62	59·5	−2·5	4·68	132	131·4	−·6
·73	72	69·3	−2·7	6·06	142	141·3	−·7
1·02	82	79·3	−2·7	7·85	152	151·6	−·4
1·42	92	89·9	−2·1	9·99	162	161·7	−·3
1·95	102	100·2	−1·8	12·64	172	171·8	−·2
2·65	112	110·7	−1·3	15·91	182	182·0	0
3·57	122	121·3	− ·7	29·80	212	212·	

The formula deviates slightly from the observations at very low pressures. Dalton says that it is next to impossible to free any liquid entirely from air; of course if any air enter, it unites its force to that of the vapour.—*Manchester Memoirs*, vol. v. p. 570. It must be recollected that according to theory the constants γ and E are the same only as long as the chemical constitution of the vapour remains the same, and they vary for different substances.

With regard to the nature of the accurate expression which connects the pressure with the temperature, opinions have hitherto been various. According to Dr. Robison, Mr. Watt found that water would distil *in vacuo* when of the temperature of 70°, and that in this case the latent heat of the steam appeared to be about 100°; and some other experiments made him suppose that the sum of the sensible and latent heats is a constant quantity. This, Dr. Robison says, is a curious and not improbable circumstance. Southern, on the contrary, concluded from experiments on the latent heat of steam at high temperatures that the *latent heat* is a constant quantity, instead of the latent heat + sensible heat being so. M. de Pambour, in speaking of Southern's view, says, "Cette opinion a paru plus rationelle à quelques auteurs, mais le première nous semble mise hors de doute par les observations que nous allons rapporter." It appears to me by no means clear that Watt entertained the opinion here attributed to him, for in a note in the Appendix to Sir David Brewster's edition of Robison's Mechanical Philosophy, vol. ii. p. 167, he professes to agree in the opinion there delivered by Southern. In p. 166 Southern records three experiments, from which he obtained 1171°, 1212°, and 1245°, for the sums of the latent + sensible heat corresponding to the temperatures or sensible heat 229°, 270°, 295°. If we take the two extreme observations, we find a difference in the sum of the latent + sensible heat of 74 degrees, corresponding to a difference in the sensible heat of 66 degrees.

If the conditions under which Laplace obtained the equation

$$V = A + B\frac{p}{\varrho}^{\frac{1}{\gamma}}$$

are admitted, the value of E different from zero shows that the absolute heat is not constant; but the preceding theory does not appear to me to furnish the means of determining the value of D, and hence of deciding with certainty whether the latent heat is constant, and whether in augmentations of heat the sensible heat only varies. I think there can be little doubt that the conditions assumed by Laplace actually obtain, and that the hypothesis attributed to Watt* must be abandoned. The experiments recorded by Mr. Parkes in the 3rd volume of the Transactions of the Society of Civil Engineers, p. 71, which show that the quantity of fuel required to evaporate a given weight of water is nearly the same whatever be the pressure of the steam, do not seem to me to authorize a different conclusion. For this is precisely what would take place if the *latent* heat be constant, and if the quantity of fuel required to generate the *latent* greatly exceed that required to generate the concomitant *sensible* heat.

The quantity γ has never before been determined for steam† or for the vapour of any liquid, properly so called, as far as I am aware. It may excite surprise that the value of γ should come out less than unity. Both Poisson and Dulong assert that it is evident that γ must surpass unity, but the reason which they assign appears to me inconclusive.

ON THE STEAM-ENGINE.

The law which connects the pressure and the temperature of steam having been unknown, various empirical rules have been given. As, however, the expressions which arise are not in a convenient form for the calculations which are required in order to ascertain the *duty* which steam-engines are capable of performing, or to solve other problems of the same nature, M. de Pambour ‡, in his work on that subject, has employed another expression, viz.

$$\mu = \frac{1}{\varrho} = \frac{1}{n + qp},$$

in which ϱ is the density of steam, p the pressure, and n and q constants. According to my expression

* Mr. Sharpe has maintained the same opinion in the 2nd vol. of the Manchester Memoirs. See Dr. Thomson's Outline of Heat and Elasticity, p. 198.

† "Quant à la valeur de γ, elle nous est jusqu'à present tout-à-fait inconnue."—Poisson, *Méc.*, tom. ii. p. 652.

‡ *Théorie de la Machine à Vapeur*, p. 111.

$$\frac{1}{\varrho} = \frac{(1-E)}{p^{\frac{1}{\gamma}} - Ep}.$$

The pressure being reckoned in atmospheres, and the density of steam corresponding to the pressure of one atmosphere (or 14·706 lbs. per square inch) being unity. If we take the density of water for unity, then as the volume of steam at the pressure of one atmosphere is 1700 times greater than that of the same weight of water,

$$\mu = -\frac{K}{p^{\frac{1}{\gamma}} - Ep} \qquad K = 1700\,(E-1)$$

$$\log K = 2{\cdot}4765041 \qquad \frac{1}{\gamma} = 1{\cdot}0134 \qquad E = 1{\cdot}17602.$$

If we suppose that a certain volume of water represented by S be transformed into vapour at the pressure p, and that M is the absolute volume of vapour which results, we shall have

$$\frac{M}{S} = \mu = \frac{[0{\cdot}4109002]}{p}\left(\frac{1}{\alpha} + \theta\right).$$

If afterwards the same volume of water is transformed into vapour at the pressure p', and that the absolute volume which the resulting vapour occupies be called M', we shall have

$$\frac{M'}{S} = \mu'$$

$$\frac{M}{M'} = \frac{p'^{\frac{1}{\gamma}} - Ep}{p^{\frac{1}{\gamma}} - Ep}$$

" Soit* P la pression *totale* de la vapeur dans la chaudière, et p' la pression qu'aura cette vapeur à son arrivée dans le cylindre, pression qui sera toujours moindre que P, excepté dans un cas particulier que nous traiterons plus loin. La vapeur pénétrera donc dans le cylindre à la pression p', et elle continuera d'affluer avec cette pression et de produire un effet correspondant, jusqu'à ce que la communication entre la chaudière et le cylindre soit interceptée. Alors il cessera d'arriver de la vapeur nouvelle dans le cylindre, mais celle qui y est déjà parvenue, commencera à se dilater pendant le reste de la course du piston, en produisant par sa détente une certaine quantité de travail, qui s'ajoutera à celle déjà produite pendant la période d'admission de la vapeur.

* The reasoning here is taken from M. de Pambour's work.

"P étant la pression de la vapeur dans la chaudière, et p' la pression qu'elle prendra à son arrivée dans le cylindre avant la détente, soit π la pression de cette vapeur en un point quelconque de la détente. Soit en même temps l la longueur totale de la course du piston, l' la portion parcourue au moment où a commencé la détente, et λ celle qui correspond au point où la vapeur a acquis la pression π. Enfin, soit encore a l'aire du piston, et c la liberté du cylindre, c'est-à-dire l'espace libre qui existe à chaque bout du cylindre, au-delà de la portion parcourue par le piston, et qui se remplit nécessairement de vapeur à chaque course; cet espace, y compris les passages aboutissants, étant représenté par une longueur équivalente du cylindre.

"Si l'on prend le piston au moment où la longueur de course parcourue est λ, et la pression π, on verra que si le piston parcourt, en outre, un espace élémentaire $d\,\lambda$, le travail élémentaire produit dans ce mouvement sera $\pi\, a\, \mathrm{d}\,\lambda$. Mais en même temps, le volume $a\,(l' + c)$ occupé par le vapeur avant la détente sera devenu $a\,(\lambda + c)$." Hence,

$$\frac{M}{M'} = \frac{p'^{\frac{1}{\gamma}} - E p'}{\pi^{\frac{1}{\gamma}} - E\pi} = \frac{\lambda + c}{l' + c}.$$

$$\lambda + c = (l' + c)\,\frac{(p'^{\frac{1}{\gamma}} - E p')}{(\pi^{\frac{1}{\gamma}} - E\pi)}$$

The elementary work produced $= \pi\, a\, \mathrm{d}\,\lambda$.

$$\int \pi\, a\, \mathrm{d}\,\lambda = \pi\, a\,(\lambda + c) - \int a\,(\lambda + c)\,\mathrm{d}\,\pi$$

$$= \pi\, a\,(\lambda + c) - \int a\,(l' + c)\,(p'^{\frac{1}{\gamma}} - E p')\,\frac{\mathrm{d}\,\varpi}{(\pi^{\frac{1}{\gamma}} - E\pi)}$$

$$= \pi\, a\,(\lambda + c) - a\,(l' + c)\,(p'^{\frac{1}{\gamma}} - E p')\int \frac{\mathrm{d}\,\varpi}{\pi^{\frac{1}{\gamma}} - E\pi}$$

$$= \pi\, a\,(\lambda + c)$$

$$+\, a\,(l' + c)\,\frac{(p'^{\frac{1}{\gamma}} - E p')\,\gamma}{E\,(\gamma - 1)}\ \log\left(1 - E\pi^{\frac{\gamma-1}{\gamma}}\right) + \text{const.}$$

This integral is to be taken from $\lambda = l'$ to $\lambda' = l$, let $\pi = p$ when $\lambda = l$, when $\lambda = l'$, $\pi = p'$.

$$\int \pi\, a\, d\lambda = p\, a\,(l + c) - p'\, a\,(l' + c)$$

$$+ a\,(l' + c)\,\frac{(p'^{\frac{1}{\gamma}} - E\,p')\,\gamma}{E\,(\gamma - 1)}\log\left\{\frac{1 - E\,p^{\frac{\gamma-1}{\gamma}}}{1 - E\,p'^{\frac{\gamma-1}{\gamma}}}\right\}$$

for the values of the constants E, γ. See p. 6.

To this must be added the work effected during the course of the piston through l', which is $p'\ a\ l$, and if R is the total pressure exerted upon unity of surface of the piston

$$p\,a\,(l + c) - p'\,a\,c$$

$$+ a\,(l' + c)\ (p'^{\frac{1}{\gamma}} - E\,p')\ \frac{\gamma}{E\,(\gamma - 1)}\log\left\{\frac{1 - E\,p^{\frac{\gamma-1}{\gamma}}}{1 - E\,p'^{\frac{\gamma-1}{\gamma}}}\right\}$$

$$= a\,R\,l \qquad \text{(A)*}$$

$R = (1 + \delta)\,r + p'' + f$, f is the friction of the machine not loaded, δ the increase of this friction due to unity of the charge r, p'' the pressure on the surface of the piston, representing the atmospheric pressure when the machine works without condensation, and otherwise the pressure of condensation in the cylinder.

If S denote the volume of water converted into vapour by the boiler in unity of time, this volume in the cylinder becomes

$$-\frac{S\,K}{p'^{\frac{1}{\gamma}} - E\,p'}$$

K being the same constant as in p. 10.

It is evident, according to the reasoning of M. de Pambour, in p. 125. of his work, that if v denote the velocity of the piston

$$\frac{S\,K}{p'^{\frac{1}{\gamma}} - E\,p'} = -\,v\,a\,\frac{l' + c}{l} \qquad \text{(B.)}$$

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* This equation is equivalent to the equation (A) of M. de Pambour, p. 123, which may be put into the form

$$p\,a\,(l + c) - p'\,a\,c + \frac{l\,S}{q\,v}\ \text{Nap. log}\left\{\frac{l + c}{l' + c}\right\} = a\,R\,l.$$

or

$$p\,a\,(l + c) - p'\,a\,c + [6{\cdot}9505960]\,\frac{l\,S}{v}\ \log\left\{\frac{l + c}{l' + c}\right\} = a\,R\,l,$$

the pressure being reckoned in lbs. per square inch.

$$p'^{\frac{1}{\gamma}} - E p' = - \frac{l S K}{a v (l' + c)} = \frac{K}{\mu'},$$

Similarly,

$$p^{\frac{1}{\gamma}} - E p = - \frac{l S K}{a v (l + c)} = \frac{K}{\mu}$$

$$\frac{1}{\mu} = \frac{l S}{a v (l + c)} \qquad \frac{1}{\mu'} = \frac{l S}{a v (l' + c)}$$

$$p a (l + c) - p' a c$$

$$- \frac{l S K \gamma}{v E (1 - \gamma)} \text{ Nap.} \log \left\{ \frac{1 - E p'^{\frac{\gamma - 1}{\gamma}}}{1 - E p^{\frac{\gamma - 1}{\gamma}}} \right\} = a R l$$

$$p a (l + c) - p' a c - (p'' + f) a l$$

$$- [4 \cdot 6411966] \frac{l S}{v} \log {}^{*} \left\{ \frac{1 - E p'^{\frac{\gamma - 1}{\gamma}}}{1 - E p^{\frac{\gamma - 1}{\gamma}}} \right\} = a l (1 + \delta) r$$

$$\log \left\{ \frac{1 - E p'^{\frac{\gamma - 1}{\gamma}}}{1 - E p^{\frac{\gamma - 1}{\gamma}}} \right\} = \log \left\{ \frac{l + c}{l' + c} \right\} - \frac{1}{\gamma} \log \left(\frac{p'}{p} \right).$$

If the machine work without expansion,

$$p = p', \qquad \log \left\{ \frac{1 - E p'^{\frac{\gamma - 1}{\gamma}}}{1 - E p^{\frac{\gamma - 1}{\gamma}}} \right\} = 0,$$

$$r = \frac{p - (p'' + f)}{1 + \delta}$$

The data upon questions relating to the steam-engine are the quantities a, l, l', S, and v, and it is evident that from these quantities the quantities μ and μ' may at once be found by an easy arithmetical operation; from these the following table will give the corresponding pressures p and p', and these pressures being introduced into equation A, the value of $a\,r$ may be easily found.

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* Log. of Briggs, the pressure being reckoned in atmospheres, the log. of the constant is [7·9670537], the pressure being reckoned in lbs. per square foot.

Table showing the volume (compared with that of water at 212°), and the temperature of steam.

Pressure in lbs. per square inch. 14·706 × p.	Temperature. Fahrenheit. τ Air*. Therm.	Temperature. Fahrenheit. τ Merc. Therm.	Volume. μ†	Pressure in lbs. per square inch. 14·706 × p.	Temperature. Fahrenheit. τ Air. Therm.	Temperature. Fahrenheit. τ Merc. Therm.	Volume. μ
1	101·5°	101·5°	20816	56	287·6°	289·5°	498
2	126·0	126·0	10871	57	288·7	290·7	490
3	141·4	141·4	7442	58	289·8	291·8	482
4	153·0	153·0	5691	59	290·9	293·0	474
5	162·2	162·2	4622	60	292·0	294·1	467
6	170·1	170·1	3902	61	293·0	295·1	460
7	176·8	176·8	3381	62	294·0	296·2	453
8	182·8	182·8	2986	63	295·0	297·2	447
9	188·3	188·3	2678	64	296·0	298·2	440
10	193·2	193·2	2429	65	297·0	299·2	434
11	197·8	197·8	2223	66	298·0	300·3	428
12	202·0	202·0	2051	67	299·0	301·3	422
13	205·9	205·9	1905	68	300·0	302·3	417
14	209·5	209·5	1778	69	301·0	303·3	411
14·706	212·0	212·0	1700	70	301·9	304·2	406
15	212·9	212·9	1669	71	302·8	305·2	401
16	216·3	216·4	1572	72	303·7	306·1	396
17	219·3	219·5	1487	73	304·6	307·0	391
18	222·3	222·6	1410	74	305·5	308·0	386
19	225·1	225·5	1342	75	306·4	308·9	381
20	227·9	228·3	1280	76	307·3	309·8	377
21	230·4	230·9	1224	77	308·2	310·7	372
22	232·9	233·5	1172	78	309·1	311·6	368
23	235·2	235·8	1125	79	310·0	312·6	364
24	237·6	238·3	1082	80	310·9	313·5	359
25	239·8	240·6	1042	81	311·8	314·5	355
26	242·0	242·8	1005	82	312·7	315·4	351
27	244·0	244·9	971	83	313·5	316·3	348
28	246·1	247·0	939	84	314·3	317·1	344
29	248·0	249·0	909	85	315·1	317·9	340
30	250·0	251·0	881	86	315·9	318·7	337
31	251·8	252·8	855	87	316·7	319·6	333
32	253·7	254·8	831	88	317·5	320·4	330
33	255·4	256·5	808	89	318·3	321·2	326
34	257·2	258·3	786	90	319·1	322·0	323
35	258·9	260·1	765	91	319·9	322·9	320
36	260·6	261·8	746	92	320·7	323·7	317
37	262·2	263·5	727	93	321·5	324·5	313
38	263·8	265·1	709	94	322·3	325·3	310
39	265·3	266·6	693	95	323·0	326·0	307
40	266·8	268·2	677	96	323·7	326·8	305
41	268·2	269·6	662	97	324·4	327·5	302
42	269·7	271·2	647	98	325·1	328·3	299
43	271·1	272·6	633	99	325·8	329·0	296
44	272·5	274·1	620	100	326·5	329·7	293
45	273·8	275·4	608	105	330·0	333·3	281
46	275·2	276·8	596	120	339·7	343·4	249
47	276·5	278·2	584	135	348·4	352·4	223
48	277·8	279·5	573	150	356·5	360·8	203
49	279·1	280·8	562	165	363·9	368·4	186
50	280·4	282·1	552	180	370·7	375·5	172
51	281·7	283·5	542	195	377·0	382·0	160
52	282·9	284·7	532	210	383·3	388·4	150
53	284·1	286·0	523	225	389·0	394·4	141
54	285·2	287·1	514	240	394·5	400·1	133
55	286·4	288·3	506				

$$* \ \tau = -\frac{[2{\cdot}0651059]}{p^{\cdot 0184} - 1{\cdot}17602} - 448^{\circ}$$

$$\dagger \ \mu = \frac{[0{\cdot}4109002]\left(\frac{1}{\alpha} + \theta\right)}{p}.$$

The following example will serve to show in what manner the table was calculated.

Ex.—Calculation of the temperature and volume of steam for the pressure of 180 lbs. per square inch.

$$\log 180 = 2{\cdot}2552725$$
$$\log 14{\cdot}706 = 1{\cdot}1674946$$

$$\log p = 1{\cdot}0877779 \times {\cdot}0134 = {\cdot}01457622386 = \log 1{\cdot}03413$$

$$\begin{array}{r} 1{\cdot}17602 \\ 1{\cdot}03413 \\ \hline \end{array} \quad 2{\cdot}0651059$$

$$\log {\cdot}14189 = 9{\cdot}1519518$$

$$2{\cdot}9131541 = \log 818{\cdot}7$$
$$448{\cdot}0$$

$$\text{Temp. Fahr. Air Therm.} = 370{\cdot}7$$
$$4{\cdot}8$$

$$\text{Temp. Fahr. Merc. Therm.} = 375{\cdot}5$$

$$0{\cdot}4109002$$
$$\log \frac{1}{p} = 8{\cdot}9122221$$
$$2{\cdot}9131541$$

$$2{\cdot}2362764 = \log 172 \qquad \mu = 172$$

The following data are taken from M. de Pambour's work on the Steam Engine, p. 238.

$l' = {\cdot}25\, l$ $\quad v = 250$ $\quad f = {\cdot}5$ (lb. per square inch) $\quad \delta = {\cdot}14$

$l = 10$ ft. $\quad a = 12{\cdot}566$ sq. ft. $\quad S = {\cdot}927$ cub. ft.

$p'' = 4$ lbs. per square inch. $\quad c = {\cdot}05\, l$

$\log v = 2{\cdot}3979400$	$\log v = 2{\cdot}3979400$
$\log a = 1{\cdot}0991971$	$\log a = 1{\cdot}0991971$
$\log {\cdot}30 = 9{\cdot}4771213$	$\log 1{\cdot}05 = 0{\cdot}0211893$
$2{\cdot}9742584$	$3{\cdot}5183264$
$\log S = 9{\cdot}9670797$	$\log S = 9{\cdot}9670797$
$3{\cdot}0071787 = \log 1016{\cdot}6$	$3{\cdot}5512467 = \log 3558{\cdot}3$
$\mu' = 1016{\cdot}6$	$\mu = 3558{\cdot}3$

Hence by the table $p' = 25{\cdot}686$, $p = 6{\cdot}627$ reckoned in lbs. per square inch.

$p' = 1{\cdot}7466$, $\quad p = {\cdot}4506$, $\quad p'' + f = {\cdot}3060$ *in atmospheres.*

$\log 1{\cdot}7466 = {\cdot}2422019$, $\qquad \log {\cdot}4506 = 9{\cdot}6538224$

${\cdot}2422019 \times {\cdot}0134 = {\cdot}00324550546$

$$\begin{array}{rl} & 9{\cdot}9967545 \\ \log E = & 0{\cdot}0704184 \\ \hline & 0{\cdot}0671729 = \log 1{\cdot}16727 \end{array}$$

$$\begin{array}{rl} 0{\cdot}3461776 \times {\cdot}0134 = & {\cdot}00463877984 \\ & {\cdot}0704184 \\ \hline & {\cdot}0750571 = \log 1{\cdot}18865 \end{array}$$

$$\begin{array}{rl} \log {\cdot}18865 = & 9{\cdot}2756568 \\ \log {\cdot}16727 = & 9{\cdot}2234181 \\ \hline & 0{\cdot}0522387 \end{array}$$

$$\begin{array}{rl} \log p = & 9{\cdot}6538224 \\ \log a = & 1{\cdot}0991971 \\ \log (l + c) = & 1{\cdot}0211893 \\ \hline & 1{\cdot}7742088 \\ & 59{\cdot}457 \end{array}
\qquad
\begin{array}{rl} \log p' = & 0{\cdot}2422019 \\ \log a = & 1{\cdot}0991971 \\ \log c = & 9{\cdot}6989700 \\ \hline & 1{\cdot}0403690 \\ & 10{\cdot}974 \\ & 38{\cdot}451 \\ \hline & 49{\cdot}425 \end{array}
\qquad
\begin{array}{rl} \log (p'' + f) = & 9{\cdot}4857179 \\ \log a = & 1{\cdot}0991971 \\ \log l = & 1{\cdot}0000000 \\ \hline & 1{\cdot}5849150 \\ & 38{\cdot}451 \end{array}$$

$$\begin{array}{rl} & 4{\cdot}6411966 \\ \log l = & 1{\cdot}0000000 \\ \log S = & 9{\cdot}9671554 \\ & 8{\cdot}7179923 \\ \hline & 4{\cdot}3263443 \\ \log v = & 2{\cdot}3979400 \\ \hline & 1{\cdot}9284043 \\ & 84{\cdot}802 \\ & 59{\cdot}457 \\ \hline & 144{\cdot}259 \\ & 49{\cdot}425 \\ \hline & 94{\cdot}834 \end{array}
\qquad
\begin{array}{rl} \log 94{\cdot}834 = & 1{\cdot}9769641 \\ \log 14{\cdot}706 = & 1{\cdot}1674946 \\ \log 144 = & 2{\cdot}1583625 \\ \hline & 5{\cdot}3028212 \\ \log l (1 + \delta) = & 1{\cdot}0569049 \\ \hline & 4{\cdot}2459163 \\ a r = & 17616 \end{array}$$

$a r = 17616$ expressed in lbs., M. de Pambour finds $a r = 17337$.

ON THE VAPOURS OF ÆTHER, ALCOHOL, PETROLEUM, AND OIL OF TURPENTINE.

The following Table is extracted from a valuable paper by Dr. Ure in the Phil. Trans. for 1818.

Table of the pressure of the vapours of æther, alcohol, petroleum or naphtha, and oil of turpentine.

Æther.	
Temp.	Pressure.
Fahr.	Inch.
34°	6·20
44	8·10
54	10·30
64	13·00
74	16·10
84	20·00
94	24·70
104	30·00
2nd.	Æther.
105°	30·00
110	32·54
115	35·90
120	39·47
125	43·24
130	47·14
135	51·90
140	56·90
145	62·10
150	67·60
155	73·60
160	80·30
165	86·40
170	92·80
175	99·10
180	108·30
185	116·10
190	124·80
195	133·70
200	142·80
205	151·30
210	166·00

Alcohol sp. gr. 0·813.	
Temp.	Pressure.
Fahr.	Inch.
32°	0·40
40	0·56
45	0·70
50	0·86
55	1·00
60	1·23
65	1·49
70	1·76
75	2·10
80	2·45
85	2·93
90	3·40
95	3·90
100	4·50
105	5·20
110	6·00
115	7·10
120	8·10
125	9·25
130	10·60
135	12·15
140	13·90
145	15·95
150	18·00
155	20·30
160	22·60
165	25·40
170	28·30
173	30·00
178·3	33·50
180	34·73
182·3	36·40
185·3	39·90
190	43·20

Alcohol sp. gr. 0·813.	
Temp.	Pressure.
Fahr.	Inch.
193·3°	46·60
196·3	50·10
200	53·00
206	60·10
210	65·00
214	69·30
216	72·20
220	78·50
225	87·50
230	94·10
232	97·10
236	103·60
238	106·90
240	111·24
244	118·20
247	122·10
248	126·10
249·7	131·40
250	132·30
252	138·60
254·3	143·70
258·6	151·60
260	155·20
262	161·40
264	166·10

Petroleum.	
Temp.	Pressure.
Fahr.	Inch.
316°	30·00
320	31·70
325	34·00
330	36·40
335	38·90
340	41·60
345	44·10
350	46·86
355	50·20
360	53·30
365	56·90
370	60·70
372	61·90
375	64·00

Oil of Turpentine.	
Temp.	Pressure.
304°	30·00
307·6	32·60
310	33·50
315	35·20
320	37·06
322	37·80
326	40·20
330	42·10
336	45·00
340	47·30
343	49·40
347	51·70
350	53·80
354	56·60
357	58·70
360	60·80
362	62·40

From these observations I take the following data for æther.

$$p = 1 \qquad \theta = 73^\circ$$

$$p' = \frac{99 \cdot 10}{30} \qquad \theta' = 143^\circ$$

$$p'' = \frac{166 \cdot 00}{30} \qquad \theta'' = 178^\circ$$

$$\frac{p''^{\beta} - 1}{p'^{\beta} - 1} = [0 \cdot 1523534].$$

Hence for æther

$$\beta = - \cdot 03153 \qquad \gamma = 1 \cdot 0325 \qquad E = \cdot 67086$$

For alcohol

$$p = 1 \qquad \theta = 141 \cdot 0^\circ$$

$$p' = \frac{97 \cdot 10}{30} \qquad \theta' = 200 \cdot 0^\circ$$

$$p'' = \frac{166 \cdot 10}{30} \qquad \theta'' = 232 \cdot 0^\circ$$

$$\frac{p''^{\beta} - 1}{p'^{\beta} - 1} = [0 \cdot 1830354].$$

$$\beta = \cdot 04025 \qquad \gamma = \cdot 96131 \qquad E = 1 \cdot 55796$$

For petroleum

$$p = 1 \qquad \theta = 284^\circ$$

$$p' = \frac{46 \cdot 86}{30} \qquad \theta' = 318^\circ$$

$$p'' = \frac{64}{30} \qquad \theta'' = 343^\circ$$

$$\frac{p''^{\beta} - 1}{p'^{\beta} - 1} = [0 \cdot 2259762]$$

$$\beta = - 0 \cdot 6268 \qquad \gamma = 1 \cdot 0668 \qquad E = \cdot 35294.$$

For oil of turpentine

$$p = 1 \qquad \theta = 272^\circ$$

$$p' = \frac{45}{30} \qquad \theta' = 304^\circ$$

$$p'' = \frac{62\cdot40}{30} \qquad \theta'' = 330^\circ$$

$$\frac{p''^{\beta} - 1}{p'^{\beta} - 1} = [0\cdot2441091].$$

$$\beta = -\cdot1816 \qquad \gamma = 1\cdot2219 \qquad E = -\cdot73937$$

And hence for the vapour of æther

$$\tau = \frac{[2\cdot2601058]}{p^{-\cdot03153} - \cdot67086} - 448^\circ.$$

For the vapour of alcohol

$$\tau = \frac{[2\cdot5396942]}{p^{\cdot04025} - 1\cdot57796} - 448^\circ.$$

For the vapour of petroleum

$$\tau = \frac{[2\cdot6940380]}{p^{-\cdot06268} - \cdot35294} - 448^\circ.$$

For the vapour of oil of turpentine

$$\tau = \frac{[3\cdot1166099]}{p^{-\cdot1816} + \cdot73937} - 448^\circ.$$

The temperature being reckoned in Fahrenheit's scale and the pressure in atmospheres.

Mr. E. Russell has calculated for me the following table, showing how far the above formulæ represent the observations of Dr. Ure. The results are exhibited in the plate annexed, and it will be seen that the discrepancies between the theory here suggested and the results of observations are chiefly owing to the irregularities of the latter, which arise doubtless from the great difficulties incidental to such experiments. When the pressures are small, the variation of temperature becomes great for a small variation of pressure, so that the agreement of theory with observation may be considered as complete, even if the absolute amount of the error of the calculated temperature is then more considerable.

Æther.			Alcohol.			Sp. gr. 0·813.			Petroleum.		
Pressure.	Temp. calc. τ	Error.	Pressure.	Temp. calc. τ	Error.	Pressure.	Temp. calc. τ	Error.	Pressure.	Temp. calc. τ	Error.
Inch.	°	°	Inch.	°	°	Inch.	°	°	Inch.	°	°
6·20	30·8	−3·2	0·40	33·6	+1·6	46·60	193·6	+0·3	30·00*	316·0	0·0
8·10	42·2	−1·8	0·56	42·8	+2·8	50·10	197·1	+0·8	31·70	320·1	+0·1
10·30	52·9	−1·1	0·70	48·1	+3·1	53·00	199·9	−0·1	34·00	325·4	+0·4
13·00	63·5	−0·5	0·86	53·3	+3·3	60·10	206·3	+0·3	36·40	330·7	+0·7
16·10	73·6	−0·4	1·00	57·2	+2·2	65·00	210·3	+0·3	38·90	335·5	+0·5
20·00	84·2	+0·2	1·23	62·6	+2·6	69·30	213·7	−0·3	41·60	340·7	+0·7
24·70	94·9	+0·9	1·49	67·8	+2·8	72·20	215·8	−0·2	44·10	345·3	+0·3
30·00*	105·0	0·0	1·76	72·4	+2·4	78·50	220·3	+0·3	46·86*	350·0	0·0
32·54	109·3	−0·7	2·10	77·4	+2·4	87·50	226·2	+1·2	50·20	355·4	+0·4
35·90	114·7	−0·3	2·45	81·9	+1·9	94·10	230·2	+0·2	53·30	360·2	+0·2
39·47	119·9	−0·1	2·93	87·3	+2·3	97·10*	232·0	0·0	56·90	365·4	+0·4
43·24	125·0	0·0	3·40	91·8	+1·8	103·60	235·6	−0·4	60·70	370·7	+0·7
47·14	129·8	−0·2	3·90	96·1	+1·1	106·90	237·4	−0·6	61·90	372·3	+0·3
51·90	135·4	+0·4	4·50	100·7	+0·7	111·24	239·8	−0·2	64·00*	375·0	0·0
56·90	140·8	+0·8	5·20	105·4	+0·4	118·20	243·3	−0·7			
62·10	145·9	+0·9	6·00	110·2	+0·2	122·10	245·2	−1·8	Oil of Turpentine.		
67·60	151·0	+1·0	7·10	116·0	+1·0	126·10	247·1	−0·9			
73·60	156·2	+1·2	8·10	120·7	+0·7	131·40	249·6	−0·1	Pressure.	Temp. calc. τ	Error.
80·30	161·6	+1·6	9·25	125·5	+0·5	132·30	250·0	0·0			
86·40	166·2	+1·2	10·60	130·5	+0·5	138·60	252·8	+0·8			
92·80*	170·8	+0·8	12·15	135·6	+0·6	143·70	255·9	+1·6	Inch.	°	°
99·10	175·0	0·0	13·90	140·8	+0·8	151·60	258·3	−0·3	30·00*	304·0	0·0
108·30	180·8	+0·8	15·95	146·3	+1·3	155·20	259·8	−0·2	32·60	310·5	+2·9
116·10	185·4	+0·4	18·00	151·1	+1·1	161·40	262·2	+0·2	33·50	312·7	+2·7
124·80	190·2	+0·2	20·30	156·1	+1·1	166·10*	264·0	0·0	35·20	316·6	+1·6
133·70	194·9	−0·1	22·60	160·7	+0·7				37·06	320·6	+0·6
142·80	199·5	−0·5	25·40	165·7	+0·7				37·80	322·2	+0·2
151·30	203·5	−1·5	28·30	170·4	+0·4				40·20	327·1	+1·1
166·00*	210·0	0·0	30·00*	173·0	0·0				42·10	330·8	+0·8
			33·50	178·0	−0·3				45·00*	336·0	0·0
			34·73	179·6	−0·4				47·30	340·0	0·0
			36·40	181·8	−0·5				49·40	343·4	+0·4
			39·90	186·1	+0·8				51·70	347·0	0·0
			43·20	189·9	−0·1				53·80	350·2	+0·2
									56·60	354·3	+0·3
									58·70	357·1	+0·1
									60·80	359·9	−0·1
									62·40*	362·0	0·0

The observations marked with an asterisk are those which were employed in procuring the constants β, E.

The æther which was observed by Dr. Ure below the pressure of 30 inch. appears to have been slightly different in quality from the other; in estimating the comparison this circumstance should be born in mind.

ON THE CONDITIONS OF THE ATMOSPHERE, AND ON THE CALCULATION OF HEIGHTS BY THE BAROMETER.

The same principles are applicable to the constitution of the atmosphere; but we are far from possessing such extensive and satisfactory data for testing the accuracy of the formulæ. The best observations for this purpose are those of M. Gay Lussac, recorded by M. Biot in the *Connaissance des Temps* for 1841, in the following table.

Table des observations par ordre de hauteurs barométriques.

Numéro des observations par ordre de pression.	Températures en degrés du thermomètre contésimal.	Moyenne des indications des deux hygromètres.	Hauteur moyenne du baromètre dans l'atmosphère, ramenée à celle d'un baromètre à niveau constant.	Hauteurs correspondantes au-dessus de l'Observatoire de Paris, calculées par la formule barométrique de M. Laplace.
			m	m
1	+30°·75	57°·5	0·76568	0·00
2	12·50	62·0	0·5381	3032·01
3	11·00	50·0	0·5143	3412·11
4	8·50	37·3	0·4968	3691·45
5	10·50	33·0	0·4905	3816·79
6	12·00	30·9	0·4666	4264·65
7	11·00	29·9	0·4626	4327·86
8			0·4528	4511·61
9	8·75	29·4	0·4528	4511·61
10	8·25	27·6	0·4404	4725·90
11	6·50	27·5	0·4353	4808·74
12	5·25	30·1	0·4229	5001·85
13	1·00	33·0	0·4141	5175·06
14	4·25	27·5	0·4114	5267·73
15	2·50	32·7	0·3985	5519·16
16	0·00	35·1	0·3918	5631·65
17	+ 0·50	30·2	0·3901	5674·85
18	− 3·00	32·4	0·3717	6040·70
19	− 1·50	32·1	0·3696	6107·19
20	− 3·25	33·9	0·3670	6143·79
21	− 7·00	34·5	0·3339	6884·14
22	− 9·50		0·3288	6977·47

I shall employ the 1st, 5th, and 21st observations for the determination of the constants, and I propose then to calculate with these constants the temperatures corresponding to the intermediate observations. As the pressures are proportional to the heights of the barometer, if the variation of gravity be neglected we may take the heights of the barometer to represent them, and we have

$p = \cdot 76568$ $\quad\theta = 30^\circ\cdot 75$ $\quad\dfrac{1}{\alpha} = 266\cdot 67$

$p' = \cdot 4905$ $\quad\theta' = 10^\circ\cdot 50$ $\quad\dfrac{1}{\alpha} + \theta' = 277\cdot 17$

$p'' = \cdot 3339$ $\quad\theta'' = -\ 7^\circ\cdot 00$ $\quad\dfrac{1}{\alpha} + \theta'' = 259\cdot 67$

I find

$$\frac{\left(\frac{p''}{p}\right)^\beta - 1}{\left(\frac{p'}{p}\right)^\beta - 1} = \frac{(\theta'' - \theta)\left(\frac{1}{\alpha} + \theta'\right)}{(\theta' - \theta)\left(\frac{1}{\alpha} + \theta''\right)}$$

$$= [0\cdot 2988164].$$

The quantity between brackets being the logarithm of the corresponding number

$$\beta = -\ \cdot 32931 \qquad \gamma = 1\cdot 4910$$

$$E = \frac{\left(\frac{p''}{p}\right)^\beta\left(\frac{1}{\alpha} + \theta''\right) - \left(\frac{1}{\alpha} + \theta\right)}{\theta'' - \theta}$$

$$= -\ 1\cdot 1618 \qquad H = \cdot 53772$$

$$\tau = \frac{[2\cdot 8081857]}{p^{-\cdot 32931} + 1\cdot 1618} - 266^\circ\cdot 67$$

in the centigrade scale, the pressure corresponding to $\cdot 76568^{\mathrm{m}}$ of mercury in the barometer being unity. In Fahrenheit's scale,

$$\tau = \frac{[3\cdot 0634582]}{p^{-\cdot 32931} + 1\cdot 1618} - 448^\circ,$$

the pressure corresponding to 30·14 inches of mercury being unity.

If we take $\gamma = 1\cdot 5$, assuming the 21st observation of M. Gay Lussac, $E = -\ 1\cdot 1920$.

The difference in the results obtained with these constants from those obtained with the other system of constants $\gamma = 1\cdot 4910$ and $E = -\ 1\cdot 1618$, is quite insignificant, only changing the density slightly in the fifth place of decimals. By taking $\gamma = 1\cdot 5$

$$\varrho' = \varrho\,(1 - q)^2\,(1 - H\,q),$$

so that the expression for the density becomes more simple, consisting of only three terms, c^{-3u}, c^{-2u}, c^{-u}, (as will be seen hereafter), which is advantageous in the theory of astronomical refractions.

Mr. Russell has calculated for me the following table, by means of my expressions, and with the constants

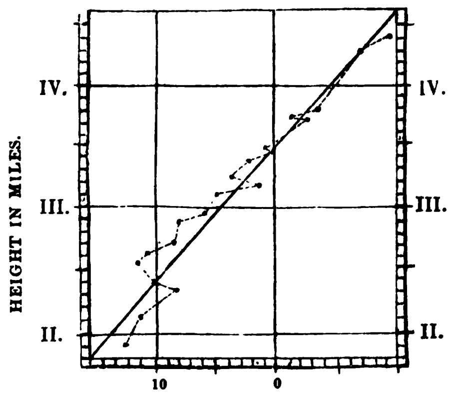
$$\gamma = 1{\cdot}5, \quad E = -1{\cdot}192:$$

No.	Observed pressure p.	Calculated height in miles.	Temperature τ.		Density ϱ.
			Calculated.	Observed.	Calculated.
1	0·76568	0·000	+30°·75	+30°·75*	·99999
2	·5381	1·895	14·74	12·50	·74276
3	·5143	2·131	12·68	11·00	·71514
4	·4968	2·310	11·10	8·50	·69473
5	·4905	2·376	10·52	10·50*	·68736
6	·4666	2·632	8·24	12·00	·65928
7	·4626	2·676	7·85	11·00	·65457
8					
9	·4528	2·785	6·87	8·75	·64299
10	·4404	2·926	5·61	8·25	·62828
11	·4353	2·985	4·99	6·50	·62222
12	·4229	3·130	3·67	5·25	·60744
13	·4141	3·236	2·80	1·00	·59692
14	·4114	3·269	2·61	4·25	·59346
15	·3985	3·428	1·05	2·50	·57818
16	·3918	3·512	0·28	0·00	·57010
17	·3901	3·533	+ 0·08	+ 0·50	·56805
18	·3717	3·772	− 2·12	− 3·00	·54576
19	·3696	3·800	− 2·37	− 1·50	·54321
20	·3670	3·835	− 2·70	− 3·25	·54004
21	·3339	4·295	− 7·00	− 7·00*	·49947
22	·3288	4·369	− 7·70	− 9·50	·49317

The observations marked with an asterisk are those which were employed in deducing the constants.

The temperatures calculated by Mr. Russell from my formula may be considered as identical with the "température régularisée par la continuité", given by M. Biot in the *Conn. des Temps*, 1841, p. 13. The observations* of M. Gay Lussac of temperature are represented in the following diagram:

* The irregularities of the observations of temperature in any future ascent might perhaps be diminished if the ballast were suffered to escape gradually in a continued stream.



The abscissa represents the temperature in degrees of the centigrade thermometer, and the ordinate the height in miles.

At first the decrements of temperature are nearly equal for equal increments of altitude. These observations by no means furnish so good a criterion of the accuracy of my formula as the observations which have been made of the temperature of steam and other vapours. The determination of the constants γ and E for the atmosphere must be repeated at some future time; for it is obvious that no great reliance can be placed upon the extreme precision of the values now obtained* until other ascents have been made, and many similar observations have been compared together. We may then hope to obtain constants accurately appertaining to a mean state of the atmosphere, and the variations which take place in their values corresponding to fluctuations of the temperature; the pressure and the humidity of the atmosphere at the earth's surface may then be investigated. M. Biot has suggested that balloons furnished with self-registering instruments should be moored over each of the principal observatories of Europe. This plan appears to me subject to great difficulties. The weight of the line attached will diminish the buoyancy, so that I apprehend it will be found impossible to send up a balloon so fastened to any considerable altitude. The escape of gas will, I imagine, render it very difficult to maintain the balloon at the same height for any length of time. The height of the balloon will also be subject to great variations from the tension of the line changing with the force and direction of the wind. I am disposed to attach much greater

* Dulong found, for atmospheric air, perfectly dry, $\gamma = 1{\cdot}421$. See Poisson, *Méc.*, tom. ii. p. 646.

value to well-regulated ascents, in which every effort should be made to reach the highest possible altitude, and of course simultaneous observations should be made at the earth's surface at such short intervals of time that every observation of the aëronaut may be comparable with a similar observation at the surface of the earth. As, however, the density and temperature of the atmosphere above the height of five miles from the earth's surface can never be the subject of direct experiment and observation, the observations which can be made upon the conditions of steam and other vapours will always maintain an indirect importance from the light which they throw upon the conditions of the atmosphere. I do not think that an examination of observations made in aërostatic ascents will ever furnish a sure guide to the relation sought between the temperature and the pressure, although if such a relation is furnished by theory and corroborated by observations of other vapours, (which can be carried through a greater extent of the thermometric scale, and, above all, through the low pressures where the variations of temperature become more rapid,) the obervations of aëronauts may serve to determine with sufficient accuracy the constants involved in the formula for atmospheric air.

The following table,* calculated by Mr. Russell, shows the density and temperature of the air at different altitudes, calculated by means of my expressions and with the constants

$$\gamma = 1{\cdot}5, \quad E = -1{\cdot}192:$$

Height in miles	Pressure p.		Temperature t.		Density ρ.
	Metres.	Inches.	Cent.	Fahr.	
0	[illegible]	[illegible]	[illegible]	[illegible]	1·00000
1	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
2	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
3	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
4	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
5	[illegible]	[illegible]	13·71	– [illegible]	[illegible]
6	[illegible]	[illegible]	[illegible]	– [illegible]	[illegible]
7	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
8	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
9	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
10	[illegible]	[illegible]	[illegible]	[illegible]	·17642
15	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
20	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
24	[illegible]	[illegible]	– [illegible]	– [illegible]	[illegible]

* In calculating this table, the law of Mariotte and Gay Lussac, expressed by the equation

$$p = \rho(1 + \alpha t)$$

has been implicitly supposed to hold good throughout; this of course is very imperfect, and it is not intended to attach precision to the temperatures assigned to the great altitudes.

As the expression which has served to calculate the temperatures evidently represents the state of the atmosphere far within the limits of the applicability of this or any other formula founded upon a state of repose to an atmosphere continually agitated by currents, it must of course serve to eliminate the density and to obtain an expression for the height in terms of the pressures and temperatures at the extremities of any atmospheric column.

If z be the altitude of the place above any fixed point, a the distance of the fixed point from the centre of the earth, g the force of gravity,

$$\frac{\mathrm{d}\,p}{\varrho} = -\frac{g\,a^2}{(a+z)^2}\,\mathrm{d}\,z,$$

and putting the expression for ϱ' at p. 3,

$$\frac{k\,(1+\alpha\,\theta)\,(p^\beta - E)\,\mathrm{d}\,p'}{p'\,(p'^\beta - E)} = -\frac{g\,a^2\,\mathrm{d}\,z'}{(a+z')^2}.$$

This expression can be integrated, and I find, supposing $z=0$, after a proper determination of the constants,

$$\frac{z'}{1+\frac{z'}{a}} =$$

$$\frac{k\,(1+\alpha\,\theta)}{g}\,\frac{\{p^\beta - p'^\beta\}\left\{\frac{1}{\alpha}+\theta'\right\}}{\beta\left\{p'^\beta\left(\frac{1}{\alpha}+\theta'\right) - p^\beta\left(\frac{1}{\alpha}+\theta\right)\right\}}\ \text{Nap. log}\left\{\frac{\left(\frac{1}{\alpha}+\theta'\right)}{\left(\frac{1}{\alpha}+\theta\right)}\left(\frac{p'}{p}\right)^\beta\right\}$$

If the variation of the force of gravity be neglected the pressures p, p' may be represented by the heights of the barometer h, h'. If M be the *modulus* or the quantity by which Naperian logarithms must be multiplied to give common logarithms, Laplace makes

$$\frac{k}{g\,M} = 18337^{\mathrm{m}}\cdot46. \qquad \log M = 9\cdot6377843.$$

In order to give an example of the use of this expression I take the 21st observation of Gay Lussac

$$h = \cdot76568 \qquad \theta = 30\cdot75$$

$$h' = \cdot3339 \qquad \theta' = -7\cdot00$$

$$\log 18337{\cdot}46 \qquad = 4{\cdot}2633392$$

$$\log (1 + \alpha\theta) \qquad = 0{\cdot}0474015$$

$$\log \left\{ \frac{(k^{\beta} - k'^{\beta})\left(\frac{1}{\alpha} + \theta'\right)}{k'^{\beta}\left(\frac{1}{\alpha} + \theta'\right) - k^{\beta}\left(\frac{1}{\alpha} + \theta\right)} \right\} \qquad = 0{\cdot}2696699$$

$$\log \left\{ \log \left\{ \frac{\left(\frac{1}{\alpha} + \theta'\right)}{\left(\frac{1}{\alpha} + \theta\right)} \left(\frac{k'}{k}\right)^{\beta} \right\} \right\} \qquad = 8{\cdot}7762776$$

$$3{\cdot}3566882$$

$$\log \beta \qquad 9{\cdot}5176049$$

$$3{\cdot}8390833$$

$$\frac{z'}{1 + \frac{z'}{a}} = 6903{\cdot}7$$

$\log a = 6{\cdot}8041168$ in metres.

$z' = 6921{\cdot}7$ metres.

If

$$\left(\frac{p'}{p}\right)^{\frac{\gamma-1}{\gamma}} = 1 - q \qquad \frac{E p^{\frac{1-\gamma}{\gamma}}}{1 - E p^{\frac{\gamma-1}{\gamma}}} = - H \text{ as before, p. 3.}$$

The expression for z' may be put into the form

$$\frac{z'}{1 + \frac{z'}{a}} = - \frac{k(1 + \alpha\theta)}{g H \beta} \text{ Nap. log } (1 - Hq).$$

If $\gamma = 1{\cdot}49138$ when $p' = 0$, $q = 1$, we get for the superior limit of the atmosphere an altitude of about 24 miles, or 38918 metres.

Ultimately the intensity of the cold deprives the air of its elasticity *. The density therefore requires in strictness to be re-

* See Poisson, *Théorie de la Chaleur*, p. 460, "On peut se représenter une colonne atmosphérique qui s'appuie sur la mer, par example, comme un fluide élastique terminé par deux liquides, dont l'un a une densité et une température ordinaires, et l'autre une température et une densité excessivement faibles." See also Biot, *Conn. de Temps*, 1841.

presented by a discontinuous function; for the formula suggested in this treatise is of course only applicable so long as the air exists in the state of an elastic vapour. The freezing point of air is unknown, and we cannot decide when this condition ceases to obtain.

Delambre estimates the height of the atmosphere as deduced from the phenomena of twilight* at 70,800 metres; but this calculation is open to objection. See *Conn. de Temps*, 1841, p. 58.

I have given the example of the calculation of a height by an observation of the barometer, in order to show how my formula for the density may be employed; but however inaccurate in principle the method in use may be, it is sufficiently exact for elevations accessible to man. In all inquiries, however, connected with the condition of the higher regions of the atmosphere, and in the various hypotheses which may be made respecting the decrement of temperature, the corresponding height must be calculated by an appropriate formula, procured agreeably to the hypothesis which may be adopted. Our information respecting the state of the higher regions of the atmosphere is I think more likely to be improved by observations made in aëronautic ascents than by those made on the sides of mountains.

Let $u = -$ Nap. $\log(1 - Hq)$ $\qquad i = \dfrac{k(1 + \alpha\theta)}{agH\beta}$

$$\frac{z'}{1 + \dfrac{z'}{a}} = aiu.$$

At the summit of the atmosphere $q = 1$, if u'' be the corresponding value of u,

$$u'' = -\text{Nap.}\log(1 - H) \qquad c^{-u''} = 1 - H,$$

c being the number of which the hyperbolic logarithm is unity.

$$\frac{Ep^{-\beta}}{1 - Ep^{-\beta}} = -H. \quad \text{See p. 3.}$$

p being the pressure at the lower station; the pressure for $\cdot 76568^{\text{m}}$ or 30·14 inches of mercury in the barometer being unity.

I get, when

$$\gamma = 1{\cdot}5 \qquad \beta = -\frac{1}{3} \qquad E = -1{\cdot}192,$$

the following formula for calculating heights by observations of the barometer:

---

* See Delambre's *Astronomie*, vol. i. p. 337, and Lalande's *Ast.*, vol. ii. art. 2270.

$$\frac{z'}{1 + \frac{z'}{a}} = [4{\cdot}7404605] \frac{(1 + \alpha\theta)}{H} \log (1 - Hq) \text{ in French metres,}$$

$$= [5{\cdot}2564585] \frac{(1 + \alpha\theta)}{H} \log (1 - Hq) \text{ in English feet,}$$

$$= [1{\cdot}5338195] \frac{(1 + \alpha\theta)}{H} \log (1 - Hq) \text{ in English miles}$$

the temperature θ at the lower station being reckoned from the freezing point.

$$\text{Log}\, \alpha = 7{\cdot}3187588 \text{ for Fahrenheit's scale.}$$

If we assume the 21st observation of Gay Lussac, and suppose $\gamma = 1{\cdot}4$, I find

$$\beta = -{\cdot}2857 \qquad E = -{\cdot}8405 \qquad \log H = 9{\cdot}6596173$$

In Fahrenheit's scale

$$\tau = \frac{[2{\cdot}99335785]}{p^{\beta} + {\cdot}8405} - 448^{\circ}.$$

Height in miles $= [1{\cdot}9885722] \log (1 - Hq)$.

If we suppose $\gamma = 1{\cdot}5$, I find

$$\beta = -{\cdot}3333 \qquad E = -1{\cdot}1920 \qquad \log H = 9{\cdot}7354232$$

$$\tau = \frac{[30{\cdot}694832]}{p^{\beta} + 1{\cdot}1920} - 448^{\circ}.$$

Height in miles $= [1{\cdot}8457978] \log (1 - Hq)$.

If we suppose $\gamma = 1{\cdot}6$, I find

$$\beta = -{\cdot}375 \qquad E = -1{\cdot}5112 \qquad \log H = 9{\cdot}7794573$$

$$\tau = \frac{[3{\cdot}1285240]}{p^{\beta} + 1{\cdot}5112} - 448^{\circ}.$$

Height in miles $= [1{\cdot}7506111] \log (1 - Hq)$.

Mr. Russell has calculated for me the following table in order to show in what manner the density and temperature of the atmosphere vary in the higher regions under these three different suppositions.

Height in miles.	$\beta = -\cdot 2857$.			$\beta = -\frac{1}{3}$.			$\beta = -\cdot 375$.			Height in miles.
	p.	τ.	ρ.	p.	τ.	ρ.	p.	τ.	ρ.	
0	1·0000	+ 87°	1·0000	1·0000	+ 87°	1·0000	1·0000	+ 87°	1·0000	0
4	·4628	+ 24	·5248	·4630	+ 24	·5248	·4631	+ 25	·5249	4
8	·1906	− 45	·2534	·1902	− 48	·2543	·1900	− 50	·2553	8
12	·0656	−122	·1076	·0645	−130	·1086	·0635	−137	·1093	12
16	·0167	−206	·0367	·0153	−223	·0365	·0141	−240	·0362	16
20	·0022	−298	·0080	·0015	−330	·0068	·0009	−361	·0055	20
24	·0000	−399	·0003							24
	Limit 25·81 miles.			23·896 miles.			22·52 miles.			

By making $\gamma = 1\cdot5$, the expression for the density becomes simplified, $\frac{1}{1-\gamma} = -2$,

$$\rho' = \frac{\rho}{H^2} c^{-u} \left\{ c^{-u} - 1 + H \right\}^2 .$$

See p. 3.

If $\frac{\rho'}{\rho} = 1 - \omega$

$$1 - \omega = \frac{1}{H^2} \left\{ c^{-3u} - 2(1-H)c^{-2u} + (1-H)^2 c^{-u} \right\}$$

It must be recollected that the difficulty of determining the densities at different altitudes, and that of determining altitudes by observations of the barometer, rest in finding the accurate law of the temperature. So that if the expression which I have here suggested for the temperature be adopted, the expression for the density, and those for finding the elevation by observations of the barometer, follow as a matter of course, and their accuracy is unquestionable.

The employment of the formula in p. 26, for calculating heights, amounts to determining the constant E from the observations themselves, and not from previous observations. But if the constants are supposed to be known, as in calculating a series of observations made under the same circumstances, it is more simple to employ the expression

$$\frac{z'}{1 + \frac{z'}{a}} = a\, i\, u.$$

The day on which M. Gay Lussac made his ascent was very

warm, and the values of γ and H determined from his observations may differ slightly from those mean values which will be obtained hereafter from more complete data. The preceding theory supposes implicitly that a given temperature at the earth's surface always corresponds in any given place to a given pressure; this, owing to the currents, the winds, and to other causes, is not the case; for the atmosphere is never in a state of repose, and its temperature and density are in a continual state of oscillation about their mean values. The constants γ and E may also be subject to variations from fluctuations in the quantity of aqueous vapour diffused through the atmosphere.

If the decrements of temperature are the same for equal increments of altitude, which observation shows is nearly the case at small elevations,

$$\theta - \theta' = A z',$$

θ being the temperature at the lower station, θ' at the upper, and z' as before, the altitude of the latter reckoned from the former,

$$1 + \alpha \theta' = 1 + \alpha (\theta - A z'),$$

and if the variation of the force of gravity be neglected

$$\frac{\mathrm{d} p'}{p'} = - \frac{g \, \mathrm{d} z'}{k \{1 + \alpha(\theta - A z')\}}$$

$$z' = \frac{(1 + \alpha \theta)}{\alpha A} \left\{ 1 - \left(\frac{p'}{p} \right)^{\frac{k \alpha A}{g}} \right\}$$

$$\varrho' = \varrho \left\{ \frac{1 + \alpha (\theta - A z')}{1 + \alpha \theta} \right\}^{\frac{g}{k \alpha A} - 1}$$

p' being the pressure at the upper station, and p at the lower.

Mr. Ivory assumes, Phil. Trans., 1838, p. 192,

$$\frac{1 + \alpha \theta'}{1 + \alpha \theta} = 1 - f \log \frac{\varrho}{\varrho'} + (f - f') \frac{\left(\log \frac{\varrho}{\varrho'}\right)^2}{1 \,.\, 2}$$

$$- (f - 2f + f'') \frac{\left(\log \frac{\varrho}{\varrho'}\right)^3}{1 \,.\, 2 \,.\, 3} + \&c.$$

But Mr. Ivory afterwards neglects the terms depending upon f', f'', &c., so that he virtually assumes

$$\frac{1+\alpha\,\theta'}{1+\alpha\,\theta} = 1 - f\left\{\log\frac{\varrho}{\varrho'} - \frac{\left(\log\frac{\varrho}{\varrho'}\right)^2}{1\,.\,2} + \&c.\right\}$$

$$= 1 - f\left\{1 - \frac{\varrho'}{\varrho}\right\} = \frac{p'\,\varrho}{p\,\varrho'}$$

$$\frac{p'}{p} = (1-f)\,\frac{\varrho'}{\varrho} + f\,\frac{\varrho'^2}{\varrho^2}$$

Mr. Ivory makes the constant $f = \frac{2}{9}$, p. 197, so that

$$\frac{p'}{p} = [9{\cdot}8908555]\,\frac{\varrho'}{\varrho} + [9{\cdot}3467875]\,\frac{\varrho'^2}{\varrho^2}$$

$$\mathrm{d}\,p' = p\,(1-f)\,\frac{\mathrm{d}\,\varrho'}{\varrho} + 2\,p\,f\,\varrho'\,\frac{\mathrm{d}\,\varrho'}{\varrho^2}$$

$$= -\,\frac{g\,\varrho'\,\mathrm{d}\,z'}{\left(1+\frac{z'}{a}\right)^2}$$

$$\frac{z'}{1+\frac{z'}{a}} = a\,i\,u^* = \frac{p\,(1-f)}{g\,\varrho}\,\log\frac{\varrho}{\varrho'} + \frac{2\,p\,f}{g\,\varrho}\left(1-\frac{\varrho'}{\varrho}\right)$$

$$= \frac{k\,(1+\alpha\,\theta)\,(1-f)}{g}\,\log\frac{\varrho}{\varrho'} + \frac{2\,k\,(1+\alpha\,\theta)\,f}{g}\left(1-\frac{\varrho'}{\varrho}\right)$$

$$= [0{\cdot}9635418]\,\log\frac{\varrho}{\varrho'} + [0{\cdot}3582881]\left(1-\frac{\varrho'}{\varrho}\right)$$

for 50° Fahr. at the lower station.

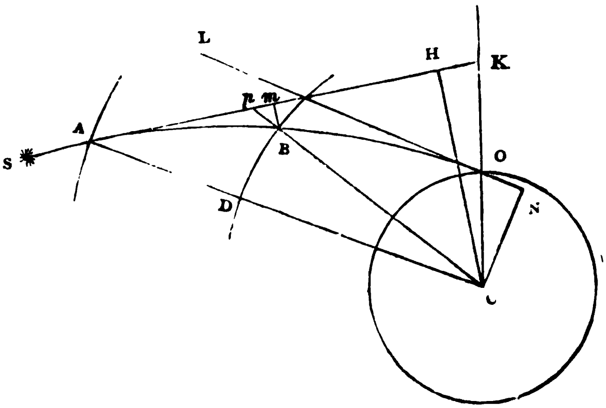
As we cannot make direct observations of the temperature and density of the highest regions of the atmosphere, it becomes very important to avail of all indirect means of investigation. The problem of Astronomical Refractions furnishes us with valuable data in this respect, and any hypothesis relative to the state of the atmosphere which will not satisfy the known phenomena of refraction must of course be discarded. In any investigation of this kind it is indispensable to employ a formula for z in terms of the density consistent with the hypothesis, which may be made respecting the decrement of temperature; it is equally indispensable to carry the integral which affords the amount of refraction through limits which are in conformity with the same supposition.

* $a\,i\,u = \sigma$ in Mr. Ivory's notation. In this page p is the pressure and ϱ is the density at the earth's surface.

ON

ASTRONOMICAL REFRACTIONS.

IF the constitution of the atmosphere be such as I have concluded, by proper substitutions in the differential equations of refraction, an accurate table of refractions is to be procured, which may be compared with that of M. Bessel obtained empirically.



Let S A O N be the trajectory described by light emanating from the star S in its passage through the atmosphere to the earth's surface at O, θ the apparent zenith distance, or the angle which the tangent to the trajectory makes with the line C O K at O, C H perpendicular to S A K, the direction of the ray before it enters the atmosphere $= y$, $a =$ C O, then

$$d . \delta \theta = - \frac{d y}{\sqrt{(a + z)^2 - y^2}}$$

$$y = a \sin \theta \sqrt{\frac{1 + 2 K \rho}{1 + 2 K \rho'}}$$

$$*\alpha = \frac{K\rho}{1 + 2K\rho} \qquad y = \frac{a \sin\theta}{\sqrt{1 - 2\alpha\omega}}$$

I assume these equations, which are proved by Mr. Ivory in the *Phil. Trans.*, 1838, and which are equivalent to similar equations given in the *Méc. Céleste.*

$$d\,.\,\delta\theta = \frac{\alpha \sin\theta\, d\omega}{(1 - 2\alpha\omega)\sqrt{\cos^2\theta + \left(\frac{z^2}{a} + \frac{z^3}{a^2}\right)(1 - 2\alpha\omega) - 2\alpha\omega}}$$

$\dfrac{z}{1 + \frac{z}{a}} = a\,i\,u.$ i being a constant and u a certain function of the density, which depends upon the constitution of the atmosphere, and which for the present may remain undefined.

$$\frac{z^2}{a} + \frac{z^2}{a^2} = 2iu + 3i^2u^2 + \&c.$$

$$d\,.\,\delta\theta = \frac{\alpha \sin\theta\, d\omega\,\{1 + 2\alpha\omega + \&c.\}}{\sqrt{\cos^2\theta + 2iu + 3i^2u^2 + \&c. - 2\alpha\omega}}$$

if $x = u - \frac{\alpha}{i}\,\omega,$ $\quad iu - \alpha\omega = ix \qquad \omega = 1 - \frac{\varrho}{\varrho'}$

$$2iu + 3i^2u^2 + \&c. - 2\alpha\omega = 2x + 3x^2.$$

"The quantities rejected being plainly of no account relatively to those retained. Further, because ω is always less than 1, $\frac{\alpha}{1 - 2\alpha\omega}$ is contained between α and $\alpha(1 + 2\alpha)$, and it may be taken equal to α, or to the mean value $\alpha(1 + \alpha)$†." Thus we have (See Phil. Trans. 1838. p. 205.)

$$d\,.\,\delta\theta = \sin\theta \times \frac{\alpha(1 + \alpha)\, d\omega}{\sqrt{\cos^2\theta + 2ix + 3i^2x^2}}$$

$$= \sin\theta \times \frac{\alpha(1 + \alpha)\, d\omega}{\sqrt{\cos^2\theta + 2ix}}$$

[1]

$$- \frac{3}{2}\,\frac{\sin\theta\,\alpha(1 + \alpha)\, i^2x^2\, d\omega}{(\cos^2\theta + 2ix)^{\frac{3}{2}}} + \&c.$$

[2]

The refraction will thus consist of two terms, which I proceed

---

* This quantity must not be confounded with the α which accompanies θ.

† Laplace introduces the same simplification. *Méc. Cél.*, vol. iv. p. 247.

to consider separately. The second term is minute, not amounting to 2″ sex. at the horizon.

Throughout this treatise on Astronomical Refractions one accent will be affixed to any symbol that it may denote the particular value of the variable which obtains at the surface of the earth, and two accents will be affixed when the particular value which obtains at the superior limit of the atmosphere is intended.

The limits of x, or x' and x'' corresponding to u' and u'', are

$x' = 0$, $x'' = u'' - \frac{\alpha}{i}$, because $\omega' = 0$, $\omega'' = 1$.

$x = u - \frac{\alpha}{i}\,\omega = u - \frac{\alpha}{i}\,\mathrm{f}\,u$; the function indicated by the letter f for the present may remain undefined.

By Lagrange's theorem

$$u = x + \frac{\alpha}{i}\,\mathrm{f}\,x + \frac{\alpha^2}{2\,i^2}\,\frac{\mathrm{d}\,(\mathrm{f}\,x)^2}{\mathrm{d}\,x} + \frac{\alpha^3}{2\,.\,3\,i^3}\,\frac{\mathrm{d}^2\,(\mathrm{f}\,x)^3}{\mathrm{d}\,x^2} + \&c.$$

$$= x + \frac{\alpha}{i}\,\omega.$$

Hence

$$\omega = \mathrm{f}\,x + \frac{\alpha}{2\,i}\,\frac{\mathrm{d}\,(\mathrm{f}\,x)^2}{\mathrm{d}\,x} + \frac{\alpha^2}{2\,.\,3\,i^2}\,\frac{\mathrm{d}^2\,(\mathrm{f}\,x)^3}{\mathrm{d}\,x^2} + \&c.$$

Let $x = x'' - X$, $u = u'' - U$, $\omega = 1 - \mathrm{F}\,U$, then

$$x'' - X = u'' - U - \frac{\alpha}{i}\,\{1 - \mathrm{F}\,U\}$$

$$X'' = 0, \quad X' = x''$$

$$X = U - \frac{\alpha}{i}\,\mathrm{F}\,U$$

$$U = X + \frac{\alpha}{i}\,\mathrm{F}\,X + \frac{\alpha^2}{2\,i^2}\,\frac{\mathrm{d}(\mathrm{F}\,X)^2}{\mathrm{d}\,X} + \&c.$$

$$\omega = 1 - \mathrm{F}\,X - \frac{\alpha}{2\,i}\,\frac{\mathrm{d}\,(\mathrm{F}\,X)^2}{\mathrm{d}\,X} - \frac{\alpha^2}{2\,.\,3\,i^2}\,\frac{\mathrm{d}^2(\mathrm{F}\,X)^3}{\mathrm{d}\,X^2} - \&c.$$

This series may be used if the atmosphere extends only to a finite altitude.

Let

$$\sqrt{\frac{\cos^2\theta + 2\,i\,x}{i}} = z \qquad \cos^2\theta + 2\,i\,x = i\,z^2$$

$$\mathrm{d}\,x = z\,\mathrm{d}\,z \qquad x^2 = \frac{i^2\,z^4 - 2\,i\,z^2\cos^2\theta + \cos^4\theta}{4\,i^2}.$$

The integral of $d . \delta \theta$ is to be taken from

$$z = \frac{\cos\theta}{\sqrt{i}} = z', \qquad \text{to } z = \frac{\sqrt{\cos^2\theta + 2\,i\,x''}}{\sqrt{i}} = z''.$$

Let $\left(\frac{d^n \omega}{d x^n}\right)'$ represent the value of $\left(\frac{d^n \omega}{d x^n}\right)$ at the former of these limits, and $\left(\frac{d^n \omega}{d x^n}\right)''$ its value at the latter, then integrating continually by parts,

$$\int \frac{\alpha \sin\theta\, d\,\omega}{\sqrt{\cos^2\theta + 2\,i\,x}} = \int \frac{\alpha \sin\theta \frac{d\,\omega}{d\,x} d\,x}{\sqrt{\cos^2\theta + 2\,i\,x}}$$

$$= \frac{\alpha \sin\theta}{\sqrt{i}} \left\{ \left(\frac{d\,\omega}{d\,x}\right)'' z'' - \left(\frac{d\,\omega}{d\,x}\right)' z' \right.$$

$$- \frac{1}{3}\left\{\left(\frac{d^2\omega}{d\,x^2}\right)'' z''^3 - \left(\frac{d^2\omega}{d\,x^2}\right)' z'^3\right\} \qquad [1]$$

$$\left. + \frac{1}{3\,.\,5}\left\{\left(\frac{d^3\omega}{d\,x^3}\right)'' z''^5 - \left(\frac{d^3\omega}{d\,x^3}\right)' z'^5\right\} \right\},$$

&c.

The second term is

$$-\frac{3\,\alpha \sin\theta\, i^2 x^2\, d\,\omega}{2\,(\cos^2\theta + 2\,i\,x)^{\frac{5}{2}}}$$

$$= -\frac{3\,\alpha\sin\theta}{8\,i^{\frac{3}{2}}}\left\{i^2 z^2 - 2\,i\cos^2\theta + \frac{\cos^4\theta}{z^2}\right\}\frac{d\,\omega}{d\,x} d\,\omega,$$

the integral of which is

$$-\frac{3\,\alpha\sin\theta}{8\,i^{\frac{3}{2}}}\left\{\frac{d\,\omega}{d\,x}\left\{\frac{i^2 z^3}{3} - 2\,i\,z\cos^2\theta - \frac{\cos^4\theta}{z}\right\}\right.$$

$$-\frac{d^2\omega}{d\,x^2}\left\{\frac{i^2 z^5}{3\,.\,5} - \frac{2\,i\,z^3\cos^2\theta}{3} - z\cos^4\theta\right\}$$

$$\left. + \frac{d^3\omega}{d\,x^3}\left\{\frac{i^2 z^7}{3\,.\,5\,.\,7} - \frac{2\,i\,z^5}{3\,.\,5}\cos^2\theta - \frac{z^3}{3}\cos^4\theta\right\}\right\},$$

$-$ &c.

$$= \frac{3\alpha\sin\theta}{S\sqrt{i}}\left\{\frac{1}{z}\left\{\frac{1}{i}\frac{d\omega}{dx}\cos^4\theta\right\}\right.$$

$$+ z\left\{-\frac{1}{i}\frac{d^2\omega}{dx^2}\cos^4\theta + z\frac{d\omega}{dx}\cos^2\theta\right\}$$

$$+ \frac{z^3}{3}\left\{\frac{1}{i}\frac{d^3\omega}{dx^3}\cos^4\theta + z\frac{d^2\omega}{dx^2}\cos^2\theta - i\frac{d\omega}{dx}\right\} \qquad [2]$$

$$+ \frac{z^5}{3.5}\left\{-\frac{1}{i}\frac{d^4\omega}{dx^4}\cos^4\theta + z\frac{d^3\omega}{dx^3}\cos^2\theta + i\frac{d^2\omega}{dx^2}\right\}$$

$$\left. + \frac{z^7}{3.5.7}\left\{\frac{1}{i}\frac{d^5\omega}{dx^5}\cos^4\theta - z\frac{d^4\omega}{dx^4}\cos^2\theta - i\frac{d^3\omega}{dx^3}\right\}\right\}$$

In order to take this quantity between the proper limits, it is only necessary to write it first with two accents and then with one accent, and take the difference of the quantities so expressed.

Instead, however, of employing the preceding expressions, I shall now introduce the auxiliary quantity e employed by Mr. Ivory. Let

$$\tan\Phi = \frac{\sqrt{2ix''}}{\cos\theta} \qquad e = \tan\frac{\Phi}{2}$$

$$\tan\Phi = \frac{2e}{(1-e^2)^2}$$

$$\sqrt{\cos^2\theta + 2ix} = \frac{\sqrt{2ix''}}{2e}\sqrt{(1-e^2)^2 + \frac{4e^2x}{x''}}$$

I assume with Mr. Ivory

$$\sqrt{(1-e^2)^2 + \frac{4e^2x}{x''}} = 1 - e^2 + 2e^2z$$

$$\frac{dx}{\sqrt{\cos^2\theta + 2ix}} = \frac{2ex''dz}{\sqrt{2ix''}}$$

then $$x = x''z - x''e^2(z - z^2).$$

Suppose $d\omega$ contains any term of the form $Ac^{-bx}dx$, then

$$\frac{\alpha\sin\theta\, d\omega}{\sqrt{\cos^2\theta + 2ix}} \text{ will contain the term}$$

$$\frac{2\alpha\sin\theta\, Aec^{-x''z} \times c^{x''e^2(z-z^2)}x''dz}{\sqrt{2ix''}},$$

which is to be integrated from $z = 0$ to $z = 1$. This integral may

be exhibited under two aspects; in the first, which is that given by Mr. Ivory, the coefficients of the different powers of e consist of a finite number of terms. In other forms of the integral, which will be given here, applicable to all atmospheres of finite altitude, the coefficients are composed of an infinite number of terms, converging with rapidity and in a form suited for numerical computation.

$$c^{-b x'' z} = 1 - b x'' z + \frac{b^2 x''^2 z^2}{1 . 2} - \&c.$$

and the single term $A c^{-b x} \mathrm{d} x$ in $\mathrm{d} \omega$ will give in

$$\int \frac{\alpha (1+\alpha) \sin \theta \, \mathrm{d} \omega}{\sqrt{\cos^2 \theta + 2 i x}} \text{ the term}$$

$$\frac{2 A \alpha (1+\alpha) \sin \theta}{\sqrt{2 i x''}} \Bigg\{ \Big\{ x'' - b x''^2 z + \frac{b^2 x''^2}{2} z^2 - \frac{b^3 x''^3}{2 . 3} z^3 + \&c. \Big\} e \, \mathrm{d} z$$

$$+ b (1 - z) \Big\{ x''^2 z - b x''^3 z^2 + \frac{b^2 x''^4 z^3}{2} - \frac{b^3 x''^5}{2 . 3} z^4 + \&c. \Big\} e^3 \, \mathrm{d} z$$

$$+ b^2 (1-z)^2 \Big\{ x''^3 z^2 - b x''^4 z^3 + \frac{b^2 x''^5 z^4}{2} - \frac{b^3 x''^6}{2 . 3} z^5 + \&c. \Big\} e^5 \, \mathrm{d} z$$

$$+ \&c. \Bigg\}$$

$$\int_0^1 z^m (1 - z)^n = \frac{n (n-1) (n-2) (n-3) \ldots\ldots 1}{(m+1) (m+2) \ldots\ldots (m+n+1)}.$$

Hence it will be seen that if $\mathrm{d} \omega$ contains any number of terms of the form $A c^{-b x} \mathrm{d} x$, the definite integral required

$$= \frac{2 (1+\alpha) \alpha \sin \theta}{\sqrt{2 i x''}} \Bigg\{ \Big\{ x'' + \frac{x''^2}{2} \left(\frac{\mathrm{d} \omega}{\mathrm{d} x}\right)' + \frac{x''^3}{2 . 3} \left(\frac{\mathrm{d} \omega^2}{\mathrm{d} x^2}\right)' + \frac{x''^4}{2 . 3 . 4} \left(\frac{\mathrm{d}^3 \omega}{\mathrm{d} x^3}\right)' + \&c. \Big\} e$$

$$+ \Big\{ \frac{x''^2}{2 . 3} \left(\frac{\mathrm{d} \omega}{\mathrm{d} x}\right)' + \frac{x''^3}{3 . 4} \left(\frac{\mathrm{d}^2 \omega}{\mathrm{d} x^2}\right)' + \frac{x''^4}{2 . 4 . 5} \left(\frac{\mathrm{d}^3 \omega}{\mathrm{d} x^3}\right)' + \&c. \Big\} e^3 \qquad [1]$$

$$+ \Big\{ \frac{x''^3}{3 . 4 . 5} \left(\frac{\mathrm{d}^2 \omega}{\mathrm{d} x^2}\right)' + \frac{x''^4}{4 . 5 . 6} \left(\frac{\mathrm{d}^3 \omega}{\mathrm{d} x^3}\right)' + \frac{x''^5}{2 . 5 . 6 . 7} \left(\frac{\mathrm{d}^4 \omega}{\mathrm{d} x^4}\right)' + \&c. \Big\} e^5$$

$$+ \&c. \Bigg\}$$

Another expression for $\delta \theta$ may be obtained in the following manner. Suppose $\mathrm{d} \omega$ contains any term

$$A_n (x'' - x)^n \, \mathrm{d} x = A_n X^n \, \mathrm{d} x$$

$$x'' - x = x_{\,\prime\prime}(1 - z)\,(1 + e^2 z)$$

$$\frac{A_n\,(x'' - x)^n\,d\,x}{\sqrt{\cos^2\theta + 2\,i\,x}} = \frac{2\,A_n\,x''^{n+1}\,(1 - z)^n\,(1 + e^2 z)^n\,e\,d\,z}{\sqrt{2\,i\,x''}}$$

$$\int_0^1 (1 - z)^n\,z^m = \frac{m\,(m-1)\,(m-2)\,\ldots\ldots\,1}{(n+1)\,(n+2)\,\ldots\ldots\,(n+m+1)}$$

n and m being whole numbers. Hence

$$\delta\,\theta = \frac{2\,\alpha\,(1+\alpha)\sin\theta}{\sqrt{2\,i\,x''}}\left[\left\{A_0\,x'' + \frac{A_1\,x''^2}{2} + \frac{A_2\,x''^3}{3} + \frac{A_3\,x''^4}{4} + \frac{A_4\,x''^5}{5} + \text{\&c.}\right\} e\right.$$

$$+ \left\{\frac{A_1\,x''^2}{2\,.\,3} + \frac{2\,A_2\,x''^3}{3\,.\,4} + \frac{3\,A_3\,x''^4}{4\,.\,5} + \frac{4\,A_4\,x''^5}{5\,.\,6} + \text{\&c.}\right\} e^3$$

$$+ \left\{\frac{2\,.\,1\,A_2\,x''^3}{3\,.\,4\,.\,5} + \frac{3\,.\,2\,A_3\,x''^4}{4\,.\,5\,.\,6} + \frac{4\,.\,3\,A_4\,x''^5}{5\,.\,6\,.\,7} + \frac{5\,.\,4\,A_5\,x''^6}{6\,.\,7\,.\,8} + \text{\&c.}\right\} e^5 \qquad [1]$$

$$+ \left\{\frac{3\,.\,2\,.\,1\,A_3\,x''^4}{4\,.\,5\,.\,6\,.\,7} + \frac{4\,.\,3\,.\,2\,A_4\,x''^5}{5\,.\,6\,.\,7\,.\,8} + \frac{5\,.\,4\,.\,3\,A_5\,x''^6}{6\,.\,7\,.\,8\,.\,9} + \frac{6\,.\,5\,.\,4\,A_6\,x''^7}{7\,.\,8\,.\,9\,.\,10} + \text{\&c.}\right\} e^7$$

$$\left. + \text{\&c.}\right]$$

A_0, A_1, A_2, &c. are constants, the numerical value of which depends upon the constitution of the atmosphere.

$$\omega = 1 - A_0\,X - \frac{A_1\,X^2}{2} - \frac{A_2\,X^3}{3} - \text{\&c.}$$

the first term is necessarily equal to unity, because when $X = 0$, $\omega = 1$, when $\omega = 0$, $X = X' = x''$, therefore generally

$$A_0\,x'' + \frac{A_1\,x''^2}{2} + \frac{A_2\,x''^3}{3} + \text{\&c.} = 1.$$

Let ω_1 be written for brevity instead of $\left(\frac{d\,\omega}{d\,u}\right)$,

ω_2 . . . . $\left(\frac{d^2\,\omega}{d\,u^2}\right)$,

ω_3 . . . . $\left(\frac{d^3\omega}{d\,u^3}\right)$,

then the quantities $\left(\frac{d\,\omega}{d\,x}\right)$, $\left(\frac{d\,\omega^2}{d\,x^2}\right)$ might be deduced from ω_1, ω_2, &c., in the following manner, without having recourse to the series

$$\omega = \mathrm{f}\,x + \frac{\alpha}{2\,i}\,\frac{\mathrm{d}\,.\,(\mathrm{f}\,x)^2}{\mathrm{d}\,x} + \frac{\alpha^2}{2\,.\,3\,i^2}\,\frac{\mathrm{d}^2\,.\,(\mathrm{f}\,x)^3}{\mathrm{d}\,x^2} + \text{\&c.}$$

Since

$$u - \frac{\alpha}{i}\,\omega = x \qquad\qquad \frac{\mathrm{d}\,x}{\mathrm{d}\,u} = 1 - \frac{\alpha}{i}\,\frac{\mathrm{d}\,\omega}{\mathrm{d}\,u} = 1 - \frac{\alpha}{i}\,\omega_1$$

$$\frac{\mathrm{d}\,\omega}{\mathrm{d}\,u} = \frac{\mathrm{d}\,\omega}{\mathrm{d}\,x}\,\frac{\mathrm{d}\,x}{\mathrm{d}\,u}$$

therefore

$$\frac{\mathrm{d}\,\omega}{\mathrm{d}\,x} = \frac{\omega_1}{1 - \frac{\alpha}{i}\,\omega_1},$$

similarly

$$\frac{\mathrm{d}^2\,\omega}{\mathrm{d}\,x^2} = \frac{\omega_2}{\left(1 - \frac{\alpha}{i}\,\omega_1\right)^3}$$

$$\frac{\mathrm{d}^3\,\omega}{\mathrm{d}\,x^3} = \frac{\omega_3}{\left(1 - \frac{\alpha}{i}\,\omega_1\right)^4} + \frac{3\,\frac{\alpha}{i}\,(\omega_2)^2}{\left(1 - \frac{\alpha}{i}\,\omega_1\right)^5}$$

$$\frac{\mathrm{d}^4\,\omega}{\mathrm{d}\,x^4} = \frac{\omega_4}{\left(1 - \frac{\alpha}{i}\,\omega_1\right)^5} + \frac{10\,\frac{\alpha}{i}\,\omega_3\,\omega_2}{\left(1 - \frac{\alpha}{i}\,\omega_1\right)^6} + \frac{15\,\frac{\alpha^2}{i^2}\,(\omega_2)^3}{\left(1 - \frac{\alpha}{i}\,\omega_1\right)^7}$$

The quantities $\frac{\mathrm{d}\,\omega}{\mathrm{d}\,X}$, $\frac{\mathrm{d}^2\,\omega}{\mathrm{d}\,X^2}$, &c., might also be deduced from $\frac{\mathrm{d}\,\omega}{\mathrm{d}\,U}$, $\frac{\mathrm{d}^2\,\omega}{\mathrm{d}\,U^2}$, &c., by similar expressions, only changing the signs of those terms which are multiplied by uneven powers of $\frac{\alpha}{i}$. I have not, however, found it convenient to have recourse to this method of obtaining the development of ω in terms of X. I have employed the series

$$\omega = 1 - \mathrm{F}\,X - \frac{\alpha}{2\,i}\,\frac{\mathrm{d}\,(\mathrm{F}X)^2}{\mathrm{d}\,X} - \frac{\alpha^2}{2\,.\,3\,i^2}\,\frac{\mathrm{d}^2\,(\mathrm{F}X)^3}{\mathrm{d}\,X^2} - \text{\&c.},$$

and I have found $\frac{\alpha}{2\,i}\,(\mathrm{F}\,X)^2$ by actual multiplication, $\frac{\alpha^2}{2\,.\,3\,i^2}(\mathrm{F}\,X)^3$ by multiplying $\frac{\alpha}{2\,i}(\mathrm{F}\,X)^2$ by $\frac{\alpha}{3\,i}$ $(\mathrm{F}\,X)$, &c. This process, though somewhat tedious, is extremely easy. As it may be carried on systematically, and the numbers follow each other, it is not liable to error.

So far all is general; it now remains to make some supposition with regard to the function $f\,u$, upon which the constitution of the atmosphere depends. If we take, as in p. 3, see also p. 24,

$$\omega = 1 - H^{\frac{1}{1-\gamma}} c^{-u} \left\{ c^{-u} - 1 + H \right\}^{\frac{1}{\gamma-1}}$$

$u = u'' - U$. See p. 35.

$$\omega = 1 - H^{\frac{1}{1-\gamma}} c^{-u''} c^{U} \left\{ c^{-u''} c^{U} - 1 + H \right\}^{\frac{1}{\gamma-1}}$$

$c^{-u''} = 1 - H$, therefore

$$\omega = 1 - \frac{(1-H)^{\frac{\gamma}{\gamma-1}}}{H^{\frac{1}{\gamma-1}}} c^{U} \left\{ c^{U} - 1 \right\}^{\frac{1}{\gamma-1}}$$

If we take, as in p.19, $\gamma = 1{\cdot}5$,

$$1 - \omega = \frac{c^{-u}}{H^2} \left\{ c^{-u} - 1 + H \right\}^2 = (1-q)^2 (1 - Hq)$$

$$f\,x = 1 - \frac{c^{-x}}{H^2} \left\{ c^{-x} - 1 + H \right\} \qquad p = p' (1-q)^3$$

$$\omega = 1 - \frac{(1-H)^3}{H^2} \left\{ c^{3U} - 2c^{2U} + c^{U} \right\},$$

$$i = \frac{k(1+\alpha\theta')}{a\,g\,H\beta}. \text{ See p. 25,} \qquad \theta = \frac{\varrho'}{p'} \left\{ \frac{1}{\alpha} + \theta' \right\} \frac{p}{\varrho} - \frac{1}{\alpha}$$

$$\frac{E p^{\frac{\gamma-1}{\gamma}}}{1 - E p^{\frac{\gamma-1}{\gamma}}} = -H. \text{ See p. 3.}$$

In page 29 I found $H = {\cdot}54378$ (from the observations of M. Gay Lussac) corresponding to the temperature 87·°35 of Fahrenheit, and to 30·145 inches of mercury in the barometer. As the uncertainty with respect to the values of γ and E appertaining to the mean state of the atmosphere makes it useless to have recourse to greater refinement, I shall now suppose that this value of H will be sufficiently exact for the temperature 50° of Fahrenheit and for 30 inches of mercury in the barometer at the earth's surface; the sequel will show that this hypothesis is admissible, and the calculation of i will stand thus: when $\gamma = 1{\cdot}5$

$$\log \frac{k}{g\,M} = 4{\cdot}2633392 \qquad \log \beta = 9{\cdot}5228787$$

$$\log M = 9{\cdot}6377843 \qquad \log H = 9{\cdot}7354232$$

$$\log (1 + \alpha\,\theta') = 0{\cdot}0159881 \qquad a = 6{\cdot}8041168$$

$$3{\cdot}9171116 \qquad 6{\cdot}0624187$$

$$6{\cdot}0624187$$

$$\log i = 7{\cdot}8546929 \qquad i = {\cdot}0071564$$

$$u'' = \text{Nap. log}\,\frac{1}{1 - H} = {\cdot}78478.$$

The following table shows the constitution of the atmosphere with this system of constants. It should be recollected that in calculating this table, as well as those in p. 22 and p. 27, the law of Mariotte and Gay Lussac,

$$p = k\,\varrho\,(1 + \alpha\,\theta),$$

is implicitly supposed to hold good at very low temperatures, which is to a certain extent conjectural. For this reason, and for the reason that we have not at present sufficient data for determining with great precision the constants γ and E, it is not intended to attach precision to the temperatures, densities and pressures given in the following table for the altitudes beyond 5 miles. The following example will serve to show how the table was calculated:

Calculation of the Pressure, Temperature, and Density for the height of 10 miles.

$$\log 10 = 1{\cdot}0000000 \qquad i = 7{\cdot}8546929$$

$$\log a = 3{\cdot}5974758 \text{ in miles} \qquad \log M = 9{\cdot}6377843$$

$$7{\cdot}4025242 \qquad 8{\cdot}2169086$$

$${\cdot}002526$$

$$7{\cdot}4025242$$

$$\log 1{\cdot}002526 = 0{\cdot}0011364 \qquad {\cdot}152925$$

$$9{\cdot}8470750 = \log (1 - H\,q)$$

$$7{\cdot}4013878 \qquad {\cdot}703194$$

$$8{\cdot}2169086 \qquad {\cdot}296806 = H\,q$$

$$9{\cdot}1844792$$

$$\log Hq = 9{\cdot}4725517$$
$$\log H = 9{\cdot}7354232$$
$$9{\cdot}7371285$$
$$\cdot 54592 = q$$
$$\cdot 45408 = 1 - q$$

$$\begin{array}{rrr}
\log(1-q) = 9{\cdot}6571324 & 9{\cdot}6571324 & 1{\cdot}2201080 \\
2 & 3 & \log p = 0{\cdot}4485185 \\
\hline
9{\cdot}3142648 & 8{\cdot}9713972 & 1{\cdot}6686265 \\
9{\cdot}8470750 & 1{\cdot}4771213 & \log \varrho = 9{\cdot}1613398 \\
\hline
9{\cdot}1613398 & 0{\cdot}4485185 & 2{\cdot}5072867 \\
\varrho = \cdot 14499 & p = 2{\cdot}81 & 321{\cdot}6 \\
 & & 448{\cdot}0 \\
\end{array}$$

$$\tau = [1{\cdot}2201080]\,\frac{p}{\varrho} - 448^\circ \qquad \tau = -\,126{\cdot}4$$

Table showing the constitution of the Atmosphere.

Height in miles.	Pressure p.	Temp. τ.	Density ϱ.
	Inch.	Fahr.	
0	30·00	+50·0	1·00000
1	24·61	35·0	·84611
2	20·07	19·5	·71294
3	16·25	+ 3·4	·59798
4	13·06	−13·3	·49903
5	10·41	30·6	·41403
10	2·81	126·4	·14499
15	·45	240·6	·03573
22·35	......	−448·0	....

According to this system of constants, the ascent for depressing Fahrenheit's thermometer 1° is about 352 feet.

If $\log \alpha$ (in sex. sec.) $= 1{\cdot}7669538$ $\qquad \alpha^* = \cdot 00028348$

$$x'' = u'' - \frac{\alpha}{i} = \cdot 74514 \qquad \log\left\{\frac{(1-H)^3}{H^2}\right\} = 9{\cdot}5066765.$$

* I have adjusted the value of α so that the mean refraction at 45° might exactly agree with that of M. Bessel.

$$\mathrm{F}\ U = \frac{(1-H)^3}{H^2}\left\{c^{3U} - 2c^{2U} + c^{U}\right\}.$$ See p. 32.

$$= [9{\cdot}5066765]\ U^2 + [9{\cdot}8077064]\ U^3 + [9{\cdot}8254352]\ U^4$$
$$+ [9{\cdot}6827677]\ U^5 + [9{\cdot}4289405]\ U^6 + [9{\cdot}0902531]\ U^7$$
$$+ [8{\cdot}6829116]\ U^8 + [8{\cdot}2178251]\ U^9 + [7{\cdot}7028036]\ U^{10}$$
$$+ [7{\cdot}1436673]\ U^{11} + [6{\cdot}5450488]\ U^{12} + [5{\cdot}9104758]\ U^{13}$$
$$+ [5{\cdot}2429802]\ U^{14} + [4{\cdot}5449962]\ U^{15} + [3{\cdot}8186763]\ U^{16} + \&c.$$

F X is found by changing U into X in the above series. Although the development of ω might be obtained by procuring the quantities $\frac{d\,\omega}{d\,X}$, $\frac{d^2\,\omega}{d\,X^2}$, &c., from $\frac{d\,\omega}{d\,U}$, $\frac{d\,\omega^2}{d\,U^2}$, through the expressions given in p. 37. I have preferred employing the series

$$d\,\omega = 1 - \mathrm{F}\,X - \frac{\alpha}{2\,i}\,\frac{d\,(\mathrm{F}X)^2}{d\,X} - \frac{\alpha^2}{2\,.\,3\,i^2}\,\frac{d^2\,(\mathrm{F}X)^3}{d\,X^2}$$

$$\frac{d\,\omega}{d\,x} = \frac{d\,\mathrm{F}\,X}{d\,X} + \frac{\alpha}{2\,i}\,\frac{d^2\,(\mathrm{F}X)^2}{d\,X^2} + \frac{\alpha^2}{2\,.\,3\,i^2}\,\frac{d^3\,(\mathrm{F}\,X)^3}{d\,X^3} + \&c.$$

By involution from the expression for F X the following were obtained :

$$\frac{\alpha}{2\,i}\,(\mathrm{F}\,X)^2 = [7{\cdot}3105250]\ X^4 + [7{\cdot}9125849]\ X^5 + [8{\cdot}2225672]\ X^6$$
$$+ [8{\cdot}3653575]\ X^7 + [8{\cdot}3901249]\ X^8 + [8{\cdot}3259417]\ X^9$$
$$+ [8{\cdot}1894155]\ X^{10} + [7{\cdot}9925434]\ X^{11} + [7{\cdot}7440827]\ X^{12}$$
$$+ [7{\cdot}4492548]\ X^{13} + [7{\cdot}1140008]\ X^{14} + [6{\cdot}7436508]\ X^{15}$$
$$+ [6{\cdot}3433889]\ X^{16} + [5{\cdot}9048724]\ X^{17} + \&c.$$

$$\frac{\alpha^2}{2\,.\,3\,i^2}\,(\mathrm{F}\,X^3) = [4{\cdot}9382822]\ X^6 + [5{\cdot}7162512]\ X^7 + [6{\cdot}1990788]\ X^8$$
$$+ [6{\cdot}5124208]\ X^9 + [6{\cdot}7058125]\ X^{10} + [6{\cdot}8073789]\ X^{11}$$
$$+ [6{\cdot}8341245]\ X^{12} + [6{\cdot}7797954]\ X^{13} + [6{\cdot}7124763]\ X^{14}$$
$$+ [6{\cdot}5779048]\ X^{15} + [6{\cdot}3758464]\ X^{16} + [6{\cdot}1424833]\ X^{17}$$
$$+ [5{\cdot}9091202]\ X^{18} + \&c.$$

$$\frac{\alpha^3}{2\,.\,3\,.\,4\,i^3}\,(\mathrm{F}\,X^4) = [2{\cdot}4411007]\ X^8 + [3{\cdot}3458338]\ X^9 + [3{\cdot}9500667]\ X^{10}$$
$$+ [4{\cdot}3853642]\ X^{11} + [4{\cdot}6996664]\ X^{12} + [4{\cdot}9204991]\ X^{13}$$
$$+ [5{\cdot}0668412]\ X^{14} + [5{\cdot}1470479]\ X^{15} + [5{\cdot}1817987]\ X^{16}$$
$$+ [5{\cdot}1502344]\ X^{17} + [5{\cdot}0588813]\ X^{18} + [4{\cdot}9792296]\ X^{19} + \&c.$$

$$\frac{a^4}{2.3.4.5\,i^4}(\mathrm{F}X^5) = [9{\cdot}8470092]\,X^{10} + [0{\cdot}1480392]\,X^{11} + [1{\cdot}5496223]\,X^{12}$$
$$+ [2{\cdot}0850470]\,X^{13} + [2{\cdot}4878504]\,X^{14} + [2{\cdot}8026972]\,X^{15}$$
$$+ [3{\cdot}0171128]\,X^{16} + [3{\cdot}1689304]\,X^{17} + [3{\cdot}2999429]\,X^{18}$$
$$+ [3{\cdot}3830969]\,X^{19} + [3{\cdot}4680509]\,X^{20} + \&c.$$

The coefficients of the different powers of X in these series become very small, but they acquire large multipliers from the successive differentiations which are required to give the corresponding terms in the expression for ω.

I find with this constitution of the atmosphere, A_n being the coefficient of X^n in the expression for $\frac{d\,\omega}{d\,x}$.

$$A_1 = {\cdot}6422$$
$$A_2 = 1{\cdot}9268 + {\cdot}0245 = 1{\cdot}9513$$
$$A_3 = 2{\cdot}6761 + {\cdot}1635 + {\cdot}0010 = 2{\cdot}8406$$
$$A_4 = 2{\cdot}4085 + {\cdot}5008 + {\cdot}0109 + {\cdot}0001 = 2{\cdot}9203$$
$$A_5 = 1{\cdot}6110 + {\cdot}9741 + {\cdot}0531 + {\cdot}0007 = 2{\cdot}6389$$
$$A_6 = {\cdot}8617 + 1{\cdot}3750 + {\cdot}1640 + {\cdot}0045 = 2{\cdot}4052$$
$$A_7 = {\cdot}3854 + 1{\cdot}5250 + {\cdot}3657 + {\cdot}0192 + {\cdot}0003 = 2{\cdot}2956$$
$$A_8 = {\cdot}1486 + 1{\cdot}3920 + {\cdot}6353 + {\cdot}0595 + {\cdot}0019 = 2{\cdot}2373$$
$$A_9 = {\cdot}0504 + 1{\cdot}0812 + {\cdot}9009 + {\cdot}1428 + {\cdot}0074 \ldots\ldots = 2{\cdot}1827$$
$$A_{10} = {\cdot}0153 + {\cdot}7322 + 1{\cdot}0335 + {\cdot}2802 + {\cdot}0229 + {\cdot}0007 \ldots\ldots = 1{\cdot}0848$$
$$A_{11} = {\cdot}0042 + {\cdot}4389 + 1{\cdot}1265 + {\cdot}4596 + {\cdot}0561 + {\cdot}0029 \ldots\ldots = 2{\cdot}6882$$
$$A_{12} = {\cdot}0010 + {\cdot}2366 + 1{\cdot}0329 + {\cdot}6852 + {\cdot}1094 + {\cdot}0090 \ldots\ldots = 2{\cdot}0741$$
$$A_{13} = {\cdot}0002 + {\cdot}1163 + {\cdot}8410 + {\cdot}8072 + {\cdot}2051 + {\cdot}0226 \ldots\ldots = 1{\cdot}9924$$
$$A_{14} = \ldots\ldots + {\cdot}0529 + {\cdot}5644 + {\cdot}8410 + {\cdot}3371 + {\cdot}0492 \ldots\ldots = 1{\cdot}8466.$$

$$\frac{d\,\omega}{d\,x} = {\cdot}6422\,X + 1{\cdot}9513\,X^2 + 2{\cdot}8406\,X^3 + 2{\cdot}9203\,X^4$$
$$+ 2{\cdot}6389\,X^5 + 2{\cdot}4052\,X^6 + \&c.$$

Hence by substituting these values of A_1, A_2, &c., in the expression for $\delta\,\theta$ given in p. 36, I find the first term in the refraction

$$= \sin\theta\,\{1132''{\cdot}8\,e + 639''{\cdot}9\,e^3 + 220''{\cdot}4\,e^5 + 60''{\cdot}5\,e^7 + 17''{\cdot}8\,e^9 + 5''{\cdot}5\,e^{11} + \&c.\} \qquad [1]$$

At the horizon $e = 1$, and this portion of the horizontal refraction $= 2076''{\cdot}9$.

The second term in the refraction is

$$-\frac{3}{2}\,\frac{\sin\theta\,\alpha\,i^2\,x^2\,\mathrm{d}\,\omega}{(\cos^2\theta + 2\,i\,x)^{\frac{5}{2}}}$$

$$= -\frac{3\,.\,4\sqrt{i}\,\alpha\sin\theta\,x''^2\,z^2\,\{1 - e^2 + e^2 z\}^2\,e^3\,\frac{\mathrm{d}\,\omega}{\mathrm{d}\,x}\,\mathrm{d}\,z}{2\sqrt{2\,x''}\,\{1 - e^2 + 2\,e^2 z\}^2}$$

$$= -\frac{3\,.\,4\sqrt{i}\,\alpha\sin\theta\,x''^2\,z^2\,e^3}{2\sqrt{2\,x''}}\,\frac{\mathrm{d}\,\omega}{\mathrm{d}\,x}\,\mathrm{d}\,z\left\{1 - 2\,z\,e^2 + \{5\,z^2 - 2\,z\}\,e^4 - \&c.\right\}$$

Suppose d ω contains any term

$$A_n\,(x'' - x)^n\,\mathrm{d}\,x$$

$$x'' - x = x''\,(1 - z)\,(1 + e^2 z)$$

$$\mathrm{d}\,.\,\delta\,\theta = -\frac{3\,.\,4\sqrt{i}\,\alpha\sin\theta\,x''^{n+2}}{2\sqrt{2\,x''}}\,A_n\,(1 - z)^n\,(1 + e^2 z)^n\,z^2\,e^3\,\mathrm{d}\,z$$

$$\left\{1 - 2\,z\,e^2 + \{5\,z^2 - 2\,z\}\,e^4 - \&c.\right\}$$

Neglecting the higher powers of e

$$\delta\,\theta = -\frac{3\,.\,4}{2}\,\frac{\sqrt{i}\,\alpha\sin\theta\,e^3}{\sqrt{2\,x''}}\left\{\frac{2\,.\,1\,A_1\,x''^3}{2\,.\,3\,.\,4} + \frac{2\,.\,1\,A_2\,x''^4}{3\,.\,4\,.\,5} + \frac{2\,.\,1\,A_3\,x''^5}{4\,.\,5\,.\,6} + \&c.\right\} \qquad [2]$$

With the same constants as before, $\gamma = 1{\cdot}5$, $H = {\cdot}54378$

$$\delta\,\theta = -[1{\cdot}3861838]\sin\theta\,e^3\left\{\frac{2\,.\,1\,A_1\,x''^3}{2\,.\,3\,.\,4} + \frac{2\,.\,1\,A_2\,x''^4}{3\,.\,4\,.\,5} + \frac{2\,.\,1\,A_3\,x''^5}{4\,.\,5\,.\,6} + \&c.\right\} \qquad [2]$$

$$= -1''{\cdot}5\sin\theta\,e^3.$$

This term thus amounts to only $1''{\cdot}5$ at the horizon; according to Mr. Ivory it does not amount to more than $1''$.

Hence, finally, the refraction is expressed by the following series :—

$$\text{Ref.} = \sin\theta\,\{1132''{\cdot}8\,e + 638''{\cdot}4\,e^3 + 220''{\cdot}4\,e^5 + 60''{\cdot}5\,e^7 + 17''{\cdot}8\,e^9 + 5''{\cdot}5\,e^{11} + \&c.\}$$

$$= \sin\theta \, \{[3{\cdot}0541728]\, e + [2{\cdot}8051475]\, e^3 + [2{\cdot}3443834]\, e^5 + [1{\cdot}7821564]\, e^7 + [1{\cdot}2501754]\, e^9 + [0{\cdot}7409070]\, e^{11} + \&c.\}$$

$$\tan\phi = \frac{[9{\cdot}0139814]}{\cos\theta} \qquad e = \tan\frac{\phi}{2}.$$

Mr. Russell has calculated a table of refractions from the above formula; and the following comparative view has been drawn up, with Bessel's table in the *Tabulæ Regiomontanæ* (which may be considered as the result of observations), with the table published annually in the *Conn. des Temps*, and with Mr. Ivory's table, recently published in the Phil. Trans. 1838, p. 224.

Tables of Mean Refractions.

Bar. 30 inch. Therm., Fahr., 50°.

App. Zenith Dist.	Mean Refractions.				App. Zenith Dist.
	Calculated.			Observed.	
	Conn. des Temps.	Ivory.	New Table*.	Tab. Reg.	
10°	10″30	10″30	10″30	10″30	10°
20	21·26	21·26	21·26	21·26	20
30	33·72	33·72	33·72	33·72	30
40	48·99	48·99	48·99	48·99	40
45	58·36	58·36	58·36	58·36	45
50	69·52	69·52	69·51	69·52	50
55	83·25	83·25	83·24	83·24	55
60	100·86	100·85	100·85	100·85	60
65	124·65	124·65	124·63	124·62	65
70	159·22	159·16	159·16	159·11	70
75	214·83	214·70	214·68	214·58	75
80	320·63	320·19	320·08	319·88	80
81	354·33	353·79	353·64	353·38	81
82	395·37	394·68	394·47	394·20	82
83	445·87	445·42	445·11	444·86	83
84	511·22	509·86	509·34	509·23	84
85	595·80	593·96	593·13	593·38	85
85½	648·34	646·21	645·15	647·10	85½
86	710·07	707·48	706·04	707·15	86
86½	783·07	779·92	778·08	777·36	86½
87	870·37	866·76	864·30	864·59	87
87½	975·89	971·93	968·84	972·21	87½
88	1105·1	1101·35	1097·26	1101·40	88
88½	1265·0	1262·6	1257·66	1265·5	88½
89	1464·9	1466·8	1461·49	1481·8	89
89½	1716·4	1729·5	1725·70	1764·9	89½
90	·	2072·6	2075·4		90

* The constitution of the atmosphere is shown by the table in p. 40.

The following table shows the errors of the table of the *Conn. des Temps*, of Mr. Ivory's table, and of my table, assuming Bessel's to be correct.

Zenith Dist.	Error of Table of Conn. des Temps.	Error of Mr. Ivory's Table.	Error of New Table.	Zenith Dist.	Error of Table of Conn. des Temps.	Error of Mr. Ivory's Table.	Error of New Table.
70°	+ ″·11	+ ″·05	+ ″·05	86°	+ 3″·12	+ ″·28	− 1″·11
75	+ ·25	+ ·12	+ ·10	86½	+ 5·71	+ 2·56	+ ·72
80	+ ·75	+ ·31	+ ·20	87	+ 5·78	+ 2·17	− ·29
81	+ ·95	+ ·41	+ ·26	87½	+ 3·68	− ·28	− 3·37
82	+ 1·17	+ ·48	+ ·27	88	+ 3·70	− ·05	− 4·14
83	+ 1·01	+ ·56	+ ·25	88½	− 0·50	− 2·90	− 7·80
84	+ 1·99	+ ·63	+ ·11	89	− 16·90	− 15·00	− 20·30
85	+ 2·42	+ ·58	− ·25	89½	− 48·50	− 35·40	− 39·20
85½	+ 1·24	− ·89	− 1·95				

I think that the discrepancies about 85½, 86, 86½, are caused by irregularities in the refractions of the *Tab. Reg.* Groombridge, who made many observations for the purpose of determining the amount of the refraction near the horizon, makes the horizontal refraction, for barometer 30 inch, and therm. Fahr. 50°, 2075″·4.* There is, however, some uncertainty respecting this quantity, and generally respecting the amount of refractions near the horizon. Upon this point see Delambre, *Ast.*, vol. i. p. 319. Mr. Ivory says "There is great probability that the horizontal refraction is very near 2070″, and does not exceed that quantity."

But for the irregularity in Bessel's table, which is clearly seen in the diagram inserted in the annexed plate, my table of mean refractions would be identical with the table of that distinguished astronomer to within 3 degrees of the horizon. It may therefore be safely concluded that the refractions, which belong to the atmosphere, constituted as I have supposed, in conformity with my theory of the heat of steam and other vapours, are consistent with observation.

The quantities denoted by u, (ϱ), s in the *Mécanique Céleste* correspond to the quantities, $i\,x$, ϱ', and $\frac{z}{a}$ of this treatise. The equation

$$s - \alpha\left[1 - \frac{\varrho}{(\varrho)}\right] = u, \quad \textit{Méc. Cél.}, \text{ vol. iv. p. 262,}$$

corresponds to the equation

* This curious coincidence, with my value of the horizontal refraction, is of course partly accidental.

$$u - \frac{a}{i}\,\omega = x \text{ of p. 34.}$$

Laplace assumes the relation between ω and x.

$$\varrho = (\varrho)\left[1 + \frac{f u}{l'}\right] c^{-\frac{u}{l'}}$$

or in the notation of this treatise

$$\varrho = \varrho' \left\{1 + \frac{if}{l'}\,x\right\} c^{-\frac{i x}{l'}}$$

$$\omega = -\frac{if}{l'}\,x\, c^{-\frac{i x}{l'}}$$

f and l' being arbitrary quantities, such that

$$f = \cdot 49042 \qquad l' = \cdot 000741816.$$

A table, similar to that which I have given in p. 43, showing the constitution of the atmosphere, which Laplace has assumed, would be instructive, and would enable us to judge of the admissibility of the conditions attributed to the higher regions of the atmosphere by that great philosopher.

In this treatise I have obtained an expression* for the altitude in terms of the pressure, founded upon the conditions of elastic vapours generally; this gives the relation between u and ω (see p. 30), from which a relation between x and u must afterwards be sought. When on the contrary the relation between ω and x is assumed (as was done by Laplace) an advantage may be gained in the calculation of the refraction, at the expense, however, of a simple and intelligible definition of the constitution of the atmosphere; and such a relation is of course also unconnected with any considerations founded upon the nature of caloric.

Mr. Ivory assumes the relation

$$\frac{p}{p'} = \frac{7}{9}\,\frac{\varrho}{\varrho'} + \frac{2}{9}\,\frac{\varrho^2}{\varrho'^2}$$

p' denoting the pressure, and ϱ' the density of the atmosphere at the earth's surface. From this relation it follows that (see p. 32)

---

* $\dfrac{z}{1 + \dfrac{z}{a}} = - a\, i \log (1 - H q).$

$$a\,i\,u = \frac{7\,k\,(1 + \alpha\,\theta')}{9\,g} \log \frac{\varrho'}{\varrho} + \frac{4\,k\,(1 + \alpha\,\theta')}{9\,g}\left(1 - \frac{\varrho}{\varrho'}\right)$$

k, α, θ' are L, β, τ', in Mr. Ivory's notation.

When u is a simple function of ω, this value of u may be substituted in the equation

$$x = u - \frac{\alpha}{i}\,\omega\,*, \qquad \text{(p. 34)}$$

and the value of ω in terms of x may be found at once by the reversion of the series.

Mr. Ivory makes $\dfrac{k\,(1 + \alpha\,\theta')}{a\,g} = i$, so that

$$x = -\log(1 - \omega) + f \log(1 - \omega) + \left\{2f - \frac{\alpha}{i}\right\}\omega$$

$$= -\log(1 - \omega) + f \log(1 - \omega) + h\,\omega.$$

This equation corresponds to the equation of Mr. Ivory

$$x = u - \lambda\,(1 - c^{-u}) - f\,\frac{\mathrm{d}\,c^{-u}\,R_2}{c^{-u}\,\mathrm{d}\,u} - f'\,\frac{\mathrm{d}^2\,c^{-u}\,R_4}{c^{-u}\,\mathrm{d}\,u^2} - \&\text{c.}$$

p. 203, when $f' = 0$. $\quad R_2 = 1 - u - c^{-u} \qquad \omega = 1 - c^{-u}$.

The table of mean refractions given by Mr. Ivory is founded upon the supposition that f', f'', &c. $= 0$.

$$\text{Let } i'\,x' = -\frac{k\,(1 + \alpha\,\theta)\,(1 - f)}{a\,g} \log(1 - \omega)$$

$$+ \left\{\frac{2\,k\,(1 + \alpha\,\theta)}{a\,g}\,f - \alpha\right\}\omega$$

and let $i' = \dfrac{k\,(1 + \alpha\,\theta)\,(1 - f)}{a\,g} = (1 - f)\,i$

$$h' = \frac{h}{1 - f} \qquad \lambda = \frac{\alpha}{i} \qquad h = 2f - \lambda$$

$$x' = -\log(1 - \omega) + h'\,\omega$$

i and h are identical with the quantities represented by those letters by Mr. Ivory, if

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* Mr. Ivory has the equivalent equation $\frac{\sigma}{a} = \frac{s}{a} + \kappa\,\omega = i\,x + \kappa\,\omega$, p. 203. Mr. Ivory's σ is $a\,i\,u$ in the notation of this treatise.

$\alpha = \cdot 0002835, \qquad i = \cdot 0012958, \qquad h = \cdot 22566, \ f = \frac{2}{9}$

then $\qquad i' = \cdot 0010078, \qquad h' = \cdot 29012.$

By Lagrange's theorem I find

$$\omega = 1 - c^{h'}\, c^{-x'} + h'\, c^{2h'}\, c^{-2x'} - \frac{3\,h'^2}{1\,.\,2}\, c^{3h'}\, c^{-3x'}$$

$$+ \frac{4^3\, h'^3}{1\,.\,2\,.\,3}\, c^{4h'}\, c^{4x'} - \&c.$$

$$\frac{d\,\omega}{d\,x'} = c^{h'}\, c^{-x'} - 2\,h'\, c^{2h'}\, c^{-2x'} + \frac{3^2\, h'^2}{1\,.\,2}\, c^{3h'}\, c^{-3x'}$$

$$+ \frac{4^3\, h'^3}{1\,.\,2\,.\,3}\, c^{4h'}\, c^{-4x'} - \&c.$$

The first part of the refraction is given by the expression

$$\alpha\,(1 + \alpha)\,\sin\theta \int_0^{\infty} \frac{\frac{d\,\omega}{d\,x'}\, d\,x'}{\sqrt{\cos^2\theta + 2\,i'\,x'}}$$

Let

$$n\left\{\frac{\cos^2\theta}{2\,i'} + x'\right\} = z^2$$

$$n\,x' = z^2 - \frac{n\cos^2\theta}{2\,i'} \qquad\qquad d\,x = \frac{2\,z\,d\,z}{n}$$

$$\int_0^{\infty} \frac{c^{-n\,x'}\, d\,x'}{\sqrt{\cos^2\theta + 2\,i'\,x'}} = \frac{2\,c^{z'^2}}{\sqrt{2\,i'}\,\sqrt{n}} \int_{z'}^{\infty} c^{-z^2}\, d\,z$$

At the horizon $\cos\theta = 0, \quad z' = 0$

$$\int_0^{\infty} \frac{c^{-n\,x'}\, d\,x'}{\sqrt{2\,i'\,x'}} = \frac{2}{\sqrt{2\,i'}\,\sqrt{n}} \int_0^{\infty} c^{-z^2}\, d\,z = \frac{\sqrt{\pi}}{\sqrt{2\,i'}\,\sqrt{n}}$$

This part of the horizontal refraction

$$= \frac{\alpha\,(1 + \alpha)\,\sqrt{\pi}}{\sqrt{2\,i'}}\left\{c^{h'} - \frac{2\,h'\,c^{2h'}}{\sqrt{2}} + \frac{3^2\,h'^2\,c^{3h'}}{1\,.\,2\sqrt{3}} - \frac{4^3\,h'^3\,c^{4h'}}{1\,.\,2\,.\,3\sqrt{4}} + \&c.\right\}$$

$$= \frac{\alpha\,(1 + \alpha)\,\sqrt{\pi}}{\sqrt{2\,i}}\left\{1 + \frac{1}{2}f + h - \frac{2\,h}{\sqrt{2}}\right.$$

$$+\frac{1.3}{2.4}f^2+\frac{3}{2}hf+\frac{h^2}{1.2}-\frac{2.3}{\sqrt{2}\,2}hf-\frac{2^2h^2}{\sqrt{2}}+\frac{3^2h^2}{\sqrt{3}\,1.2}$$

$$+\frac{1.3.5}{2.4.6}f^3+\frac{3.5}{2.4}hf^2+\frac{5}{1.2.2}h^2f+\frac{h^3}{1.2.3}$$

$$-\frac{2.3.5}{\sqrt{2}\,2.4}hf^2-\frac{2^2.5}{\sqrt{2}\,2}h^2f\quad\frac{2^3h^3}{\sqrt{2}\,1.2}$$

$$+\frac{3^2.5}{\sqrt{3}\,2.2}h^2f+\frac{3^3}{1.2\sqrt{3}}h^3-\frac{4^3}{1.2.3\sqrt{4}}h^3$$

$$+\frac{1.3.5.7}{2.4.6.8}f^4+\frac{3.5.7}{2.4.6}hf^3+\frac{5.7}{1.2.2.4}h^2f^2+\frac{7h^3f}{1.2.3.2}+\frac{h^4}{1.2.3.4}$$

$$-\frac{2.3.5.7}{\sqrt{2}\,2.4.6}hf^3-\frac{2^2.5.7}{\sqrt{2}\,2.4}h^2f^2-\frac{2^3.7}{\sqrt{2}\,1.2.2}h^3f$$

$$-\frac{2^4h^4}{\sqrt{2}\,1.2.3}+\frac{3^2.5.7}{1.2\sqrt{3}.2.4}h^2f^2+\frac{3^3.7}{1.2\sqrt{3}\,2}h^3f+\frac{3^4h^4}{1.2\sqrt{3}\,1.2}$$

$$-\frac{4^3.7}{1.2.3\sqrt{4}.2}h^3f-\frac{4^4}{1.2.3.4\sqrt{4}}h^4+\frac{4^5}{1.2.3.4.5\sqrt{5}}h^4+\&c.$$

$$=\frac{\alpha(1+\alpha)\sqrt{\pi}}{\sqrt{2i}}\left\{1+\lambda(\sqrt{2}-1)-f\left(2\sqrt{2}-\frac{5}{2}\right)\right.$$

$$\left.+h^2\left\{\frac{1}{2}-2\sqrt{2}+\frac{3}{2}\sqrt{3}\right\}-\frac{3}{2}hf(\sqrt{2}-1)+\frac{3}{8}f^2\right\}$$

when the higher powers of f and h are rejected, and this expression agrees with that given by Mr. Ivory, Phil. Trans., 1838, p. 207.

$$\frac{\alpha(1+\alpha)\sqrt{\pi}}{\sqrt{2i}}=2036''\cdot5$$

In atmospheres which extend to an infinite distance m (or x'' in the notation of this treatise) is infinite and e always $=1$, so that in this case the method employed by Mr. Ivory in p. 211 of his memoir, Phil. Trans., 1838, would seem at least to require further elucidation. Mr. Ivory has avoided this consideration, which would otherwise arise with the atmosphere which he has assumed, by imposing an arbitrary limit to the altitude of his atmosphere, while, however, if I am not mistaken, upon his own assumption, the density and the pressure are still finite. When n is large the numerators of the separate quantities of which the quantity A_{2n+1} in p. 211 is composed become large also.

I do not find in Mr. Ivory's paper any remarks tending to prove that the quantities which he has discarded depending upon the higher powers of f and h are incapable of producing any sensible effect; taken separately they are by no means insignificant. Nor do I think it follows as a matter of course, even if the positive and negative terms are numerically of equal value at the horizon, and so fortunately cut one another out, that the same thing will happen necessarily at all other altitudes. Unless the approximation is pushed so far as to secure the retention of all the sensible terms, or those which fairly come within the limits of the errors of observation, any comparison of the result with the valuable table of M. Bessel is illusory and only calculated to lead to incorrect conclusions. It is also indispensable that the relation implied or expressed between z and ω should be in exact conformity with the conditions attributed to the atmosphere, and in this respect the table of mean refractions of the late Mr. Atkinson in the Memoirs of the Astronomical Society appears to me not to rest upon a solid foundation.

Mr. Ivory connects the pressure and the density by the relation

$$\frac{p}{p'} = \cdot 77777 \frac{\varrho}{\varrho'} + \cdot 22222 \frac{\varrho^2}{\varrho'^2}.$$

M. Biot finds

$$\frac{p}{p'} = \cdot 761909002718 \frac{\varrho}{\varrho'} + \cdot 238167190564 \frac{\varrho^2}{\varrho'^2}$$
$$- \cdot 000076193282,$$

when the coefficients are so taken as to apply as nearly as the question will admit of throughout the whole extent of the atmosphere. But, by a careful examination of the data, M. Biot finds that at the earth's surface the following relation is more accurate.

$$\frac{p}{p'} = \cdot 956643870584 \frac{\varrho}{\varrho'} + \cdot 120146052460 \frac{\rho^2}{\rho'^2}$$
$$- \cdot 076789923044 \qquad \text{(p. 69.)}$$

and at the upper limit of the atmosphere

$$\frac{p}{p'} = \cdot 6604978157646 \frac{\rho}{\rho'} + \cdot 4159581823536 \frac{\rho^2}{\rho'^2}$$
$$- \cdot 00006605394115.$$

According to my view this equation does not contain the true mathematical law which connects the density and pressure, but of

course a parabola of this kind may always be found which will osculate the true curve at any given point.

In the *Comptes Rendus des Séances de l'Académie des Sciences*, tom. viii. p. 95, M. Biot verified and adopted a calculation of Lambert, who found from the phenomena of twilight the altitude of the atmosphere (hauteur des dernières particules d'air réfléchissantes) to be 29·115 metres.

It is unnecessary to dwell any further at present upon this subject, because if my theory of the Heat of Vapours be correct, the calculation of Astronomical Refractions, founded upon conditions which are not in conformity with that theory, becomes a problem of mere curiosity.

THE END.

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